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NUMERICAL METHODS OF INTEGRATING FUNCTIONS OF METROLOGICAL RELIABILITY OF MEASURING INSTRUMENTS

The methods used in the integration of discrete-continuous functions of metrological reliability of measuring instruments (MI) are determined. The mathematical apparatus of recursive reliability functions is given, the peculiarities of integration of these functions are determined. Algorithms for implementing and estimating the accuracy of quadrature functions in calculating the metrological reliability of MI are considered.
Keywords: *metrological reliability, recursive functions, methods of numerical integration, accuracy of quadratures.*

Визначаються методи які застосовуються при інтегруванні дискретно-безперервних функцій метрологічної надійності засобів вимірювань (ЗВ). Приводиться математичний апарат рекурсивних функцій надійності, визначаються особливості інтегрування даних функцій. Розглядаються алгоритми реалізації і оцінки точності функцій квадратур при обчисленні метрологічної надійності ЗВ.
Ключові слова: *метрологічна надійність, рекурсивні функції, методи чисельного інтегрування, точність квадратур.*

Problem's formulation

The main mathematical apparatus of the theory of metrological reliability is the theory of random functions, probability theory and mathematical statistics. In the a priori analysis of reliability allow fully defined probabilistic characteristics of reliability. Establishing an analytical expression for the distribution of random variables allows you to determine the required reliability. The choice of theoretical model of failures determines the accuracy of quantitative estimates of reliability indicators. The distribution function, which is used as a model of failures, solves the following problem — the calculation of failure rates. The classical method of calculating the reliability of systems is to determine the characteristics of indicators of system reliability based on the use of fundamental theorems of probability theory. The probabilistic-physical method is based on the use of the probabilistic-physical model and considers many states of the system with continuous time. At the same time there is a problem of integration of discrete-continuous functions of metrological reliability.

Analysis of recent research and publications

Mathematical and physical (in the sense of statistical distributions) failure distribution functions are universal and have an advantage over the classical integral distribution laws, but due to the fact that these functions are expressed through the Laplace integral and the integral of the form $\int e^{-t^2} dt$ is not expressed through elementary functions, there are difficulties in obtaining both accurate and approximate analytical solution of a definite integral for functions $p_{\neg м,k}(t)$ — probability of MI operation without metrological failure and $p_{м,k}(t)$ — work of MI with metrological refusal at calculation t_1 — time of faultless work. Application of some standardized approximations [1] of the function $\dot{\Phi}(z) = \frac{1}{2\pi} \int\limits_{-\infty}^{z} e^{-\frac{z^2}{2}} dz$ allows you to approximate these functions in a limited range of values. Decomposition into a McLaren series, or trigonometric series, and finding the partial sum of a series, finding the initial approximating function, and applying the integration apparatus in parts lead to cumbersome and inefficient calculations. Therefore, these techniques cannot be used to solve this class of problems. For example, the decomposition of the function $f(x) = 0.5 \cdot (1 + erf(x))$ in the McLaren series allows to approximate the studied function in a limited range of values $x \in [-2,2]$ in the neighborhood $x = 0$.

The use of polynomial approximation methods does not allow to approximate the function with high accuracy $f(x)$ (Fig.1), but the superposition (convolution) of the product $\{ (1 - F_{DM\,м}(t)) R_я(t) \}$ of the subintegral function can be approximated with the minimum deviation (Fig. 2).

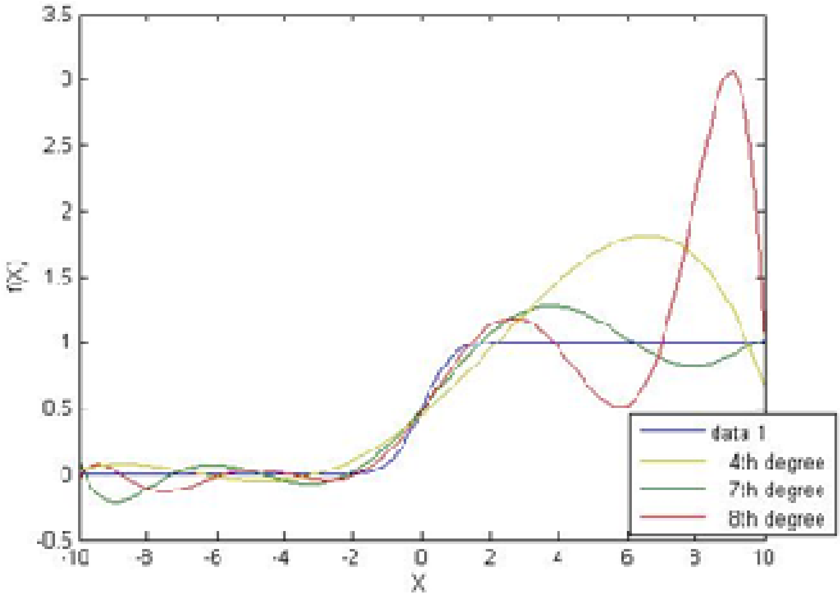

Fig. 1. Polynomial approximation by polynomials of 4th, 7th, 8th degree

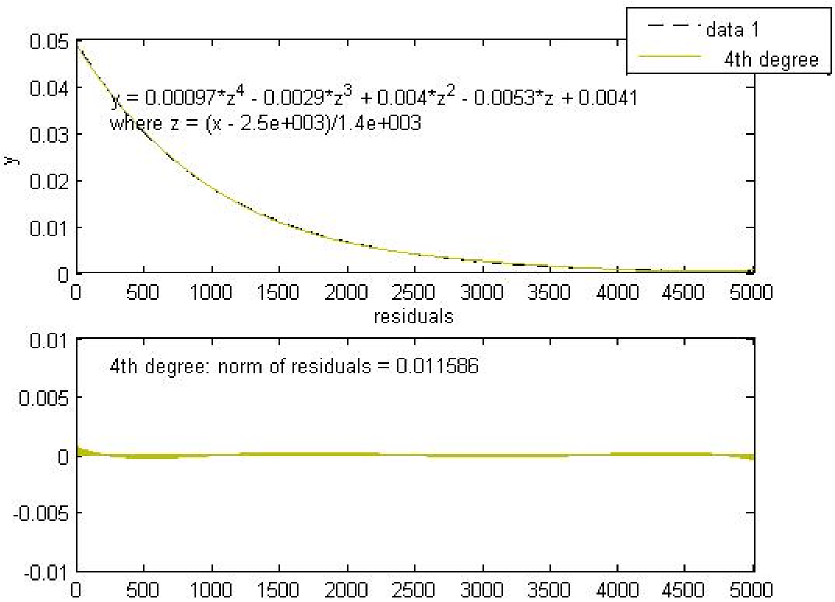

Fig. 2. Results of approximation of subintegral function $\{(1 - F_{DM\,м}(t)) R_я(t)\}$ by polynomial of the 4th order and norm of deviation of result

The ineffectiveness of these approaches is explained as follows. These expressions are functions of many variables that depend on both time and the parameters of the metrological control which in turn are parameters of the laws of distribution of probability of failure (and some are set by the target production system); when varying these parameters, the shape of the approximating functions changes and it is necessary to calculate new polynomial coefficients for each case and obtain an approximate analytical solution of the integral to find the mathematical expectation of the total metrological control time without failures.

Formulation of the study purpose

The aim of the article is to define analytical expressions for recursive functions of metrological reliability of MI. Taking into account the peculiarities of the subintegral function of the faultless operation of the MI, the substantiation of the choice and comparison of numerical integration methods, as well as the calculation of the accuracy of the algorithms for the implementation of quadrature functions.

Presenting main material

We write the expressions for the functions $p_{\neg м,k}(t)$ and $p_{м,k}(t)$, for the diffusion monotone distribution of the probabilities of metrological failures and for the exponential model of explicit failures in the following form

$$\left.\begin{aligned} p_{\neg м,k}(t)_{DM} &= p_{1DM,k}\cdot(1-F_{DM\,м}(t))R_я(t);\\ p_{м,k}(t)_{DM} &= [p_{2DM}+p_{1DM,k}\cdot F_{DM\,м}(t)]R_я(t).\end{aligned}\right\}, \tag{1}$$

where $F_{DM\,м}(t), R_я(t)$ — the functions of the probabilities of metrological failure and work without explicit failures of the metrological control in the time interval $\tau_n+kT_n\le t<\tau_n+(k+1)T_n$, respectively(verification time, verification number, verification period).

Recursive functions of probabilities of finding MI in states 1 and 2 (1 — MI is workable and used for its intended purpose; 2 — MI is used with metrological refusal) after k — its verification

$$\left.\begin{aligned} p_{1DM,k+1} &= p_{1DM,k}\cdot(1-F_{DM\,м}(T_n))R_я(T_n)(1-\alpha_n);\\ p_{2DM,k+1} &= [p_{2DM,k}+p_{1DM,k}\cdot F_{DM\,м}(T_n)]R_я(T_n)\beta_n,\end{aligned}\right\}; \tag{2}$$

for zero verification:

$$\left.\begin{aligned} p_{1DM,0} &= (1-\beta_p)(1-F_{DM\,м}(\tau_n))R_я(\tau_n)(1-\alpha_n);\\ p_{2DM,0} &= [\beta_p+(1-\beta_p)F_{DM\,м}(\tau_n)]R_я(\tau_n)\beta_n.\end{aligned}\right\} \tag{3}$$

(probability of error of the 1st kind, probability of error of the 2nd kind).

For convenience of calculations we will enter the following relative sizes: $x=\dfrac{t}{T_м}$, $x_n=\dfrac{T_n}{T_м}$, $x_\tau=\dfrac{\tau_n}{T_м}$ (work on metrological refusal). Let's denote by $Z_t=\dfrac{x-1}{v\sqrt{x}}$, $Z_n=\dfrac{x_n-1}{v\sqrt{x_n}}$, $Z_\tau=\dfrac{x_\tau-1}{v\sqrt{x_\tau}}$ (coefficient of variation). As a result, the following relations are valid for expressions (1—3)

$$\left.\begin{aligned} &F_{DM\,м}(t)=\frac{1}{2}\left[1+erf\left(\frac{Z_t}{\sqrt{2}}\right)\right];\ F_{DM\,м}(T_n)=\frac{1}{2}\left[1+erf\left(\frac{Z_n}{\sqrt{2}}\right)\right];\\ &F_{DM\,м}(\tau_n)=\frac{1}{2}\left[1+erf\left(\frac{Z_\tau}{\sqrt{2}}\right)\right];R_я(t)=e^{-\frac{t}{T_я}};R_я(T_n)=e^{-\frac{T_n}{T_я}};R_я(\tau_n)=e^{-\frac{\tau_n}{T_я}}.\end{aligned}\right\}. \tag{4}$$

The parameter is the time for explicit refusal.

According to formula (4) we write the expression t_1 for the DM-distribution of operating time on the metrological failure. To do this, introduce the following substitutions: denote by $U_t=\dfrac{1-x}{v\sqrt{x}}$, then the multiplier $(1-F_{DM\,м}(t))=\dot{\Phi}(U_t)=R_{DM\,м}(t)$ — the probability of MI operation without metrological failures at the time t.

Given the behavior of reliability functions depending on the parameters of the metrological control and the results of the integration procedures of the above quadratures as the optimal algorithm for calculating the function: $f(t)_D = \int\limits_{\tau_{n+T_n}}^{T_n+kT_n} (R_{D\,м}(t)R_я(t))dt$, which is valid for both diffusion models (index D) the adaptive Gauss-Lobatto quadrature is chosen with an accuracy of 10^{-6}.

To verify the validity of the obtained solutions using numerical integration methods to calculate the value t_1, we compared the results obtained for the adaptive Simpson and Gauss-Lobatto quadrature in the input diffusion models of metrological failures. Studies conducted for different values of metrological control parameters using a discrete-continuous operation model have shown the advantages of quadrature of Gaussian algebraic accuracy (reliability parameters for DM-distribution) $T_я = T_м = 10000 год.,\ v_м = 1$.

In the process of research and configuration of the computational procedure of the MI reliability analysis module, a library of numerical quadrature programs was created. It includes the adaptive Simpson algorithm (degree of accuracy) and programs that implement three-point and ten-point Gauss-Legendre methods. Initially, these algorithms were implemented and tested as Matlab file functions for comparison with standard Matlab numerical integration procedures, which are based on algorithms developed by computer mathematics specialists Prof. Walter Gautschi and Walter Gander [2].

When choosing a quadrature method in addition to the behavior of this function, you need to consider the accuracy and speed of numerical solutions. Since this function is quite smooth and has (albeit cumbersome) expressions for high-order derivatives, we explore the possibility of applying quadratures based on the Gauss-Legendre method.

On the basis of the algorithms given in the works carried out in [3—5], adaptive quadratures based on 3 and 10-point interpolation of the Gaussian method are implemented. It is proposed to choose the following recursive algorithm as a basis for software implementation

$$S(X,Y,f)=abs\left(\frac{[Gauss(X,Z,f)+Gauss(Z,Y,f)]-Gauss(X,Y,f)}{Gauss(X,Y,f)}\right)$$

$$gauss\_m(X,Z,f)=[S(X,Y,f)<D\rightarrow Gauss(X,Y,f),$$

$$else$$

$$gauss\_m(X,Z,f)+gaus\_m(Z,Y,f);],$$

where $G(X, Y, f)$ is the function for calculating the approximate value of the integral $\int\limits_X^Y f(x)dx$, m — selected quadrature method, D — integration threshold; first calculated $G(X,Y,f)$, $G(X,Z,f)$ and $G(Z,Y,f)$, where $Z=0{,}5(X+Y)$.

The results of calculations and comparison of the accuracy of numerical integration functions for a set of test algebraic functions by Gauss-Lobatto (quadl) and adaptive Simpson quadrature (quad) — standard Matlab 7 functions, and author-modified procedures that implement three-point (gauss_3) and ten) Gauss-Legendre method. The criterion of accuracy (given relative error) in these examples is 10^{-6}.

Comparison of the results of calculation of reliability indicators using Gauss-Legendre and Gauss-Lobatto algorithms showed that for engineering calculations it is recommended to use algorithms based on 3 and 10-point Gauss-Legendre method, and for Gauss-Lobatto research (tabl. 1, 2) [5].

Table 1. Comparison of the accuracy of quadrature algorithms

Quadrature functions	Test results on control examples		
	$I_1 = \int_0^4 13(x - x^2)e^{\frac{-3x}{2}}$	$I_2 = \int_1^6 2 + \sin(2\sqrt{x})dx$	$I_3 = \int_0^{\frac{\pi}{2}} (x^2 + x + 1)\cos(x)dx$
quadl	-1.548788372527948	8.183479207654349	2.038197427572023
quad	-1.548788476684941	8.183479195675837	2.038197433763133
gauss_3	-1.548788002258859	8.183479974070949	2.038198170813146
gauss_10	-1.548788422369362	8.183478567673465	2.038198060499660

Table 2. Comparison of the accuracy of Gaussian quadratures in calculating the integral

Coefficient of variation $v_м$	$I = \int_{\tau_n + T_n}^{T_n + kT_n} R_D(t)R_{\text{я}}(t)dt$	
	DM – metrological failure model	
	$\varepsilon_{m.gauss\_3}$	$\varepsilon_{m.\,gauss\_10}$
0,2	5.2189e-004	2.7016e-007
0,4	-0.0033	2.6547e-007
0,7	5.3990e-004	3.9050e-007
1,2	-0.0033	6.0549e-007

Conclusions

The study of the accuracy of quadratures was carried out taking into account the change in the behavior of the function at possible values of the coefficient of variation, which allows to assess the accuracy and reliability of the developed algorithms when calculating reliability. To reduce the errors of the numerical result, it is recommended to increase the accuracy of the integration threshold and the limits of calculation of recursive series of the discrete-continuous model.

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ЧИСЕЛЬНІ МЕТОДИ ІНТЕГРУВАННЯ ФУНКЦІЙ МЕТРОЛОГІЧНОЇ НАДІЙНОСТІ ЗАСОБІВ ВИМІРЮВАНЬ

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Реферат

Математичним апаратом теорії метрологічної надійності є теорія випадкових функцій, теорія ймовірності і математична статистика. Встановлення аналітичного виразу функцій розподілення випадкових величин дозволяє визначити необхідні показники надійності. Вибір теоретичної моделі відмов визначає точність кількісних оцінок показників надійності. Функція розподілу, яка застосовується в якості моделі відмов вирішує наступну задачу – розрахунок показників безвідмовності. Метод розрахунку надійності систем полягає в визначенні характеристик показників надійності систем заснований на використанні фундаментальних теорем теорії ймовірності. Ймовірністно-фізичний метод заснований на використанні ймовірністно-фізичної моделі і розглядає множину станів системи з безперервним часом. При цьому виникає проблема інтегрування дискретно-безперервних функцій метрологічної надійності.

В статті розглянуті обчислювальні методи інтегрування дискретно-безперервних функцій метрологічної надійності. Приведено рекурсивні функцій математичної теорії надійності і методи інтегрування даних функцій. Розроблені алгоритми і проведено порівняльний аналіз з оцінкою точності функцій квадратур з врахуванням зміни поведінки функції при можливих значеннях коефіцієнта варіації. При цьому враховується швидкість операцій чисельного розв'язку. Для зменшення похибок чисельного результату рекомендується збільшувати точність порогу інтегрування і границі обчислення рекурсивних рядів дискретно-безперервної моделі. Для інженерних розрахунків рекомендується застосовувати алгоритми на основі методу Гаусса-Лежандра, а для наукових досліджень Гаусса-Лобатто.

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