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GENERALIZED METHOD OF SOLUTION OF MAGNETOSTATICS EQUATION

Combinations of the method of solving the non-linear discretized leveling of magnetostatics for space-integrated levels in terms of bad efficiency or the complexity of the iterative process have been proposed. For which the method of block iterations is used, which can widen the area of stagnation, the lower iteration method, which is stable for the perfection of the non-linear alignment of magnetostatics.
***Keywords**: magnetostatics, integrated equalization, approximation, numerical modeling.*

Запропоновано комбінований метод вирішення нелінійного дискретизованого рівняння магнітостатики для просторових інтегральних рівнянь в разі розбіжності або поганої збіжності ітераційного процесу. Для цього застосовується метод поблочних ітерацій, що має ширшу область застосування, ніж метод ітерації, який застосовувався для вирішення нелінійного рівняння магнітостатики.
***Ключові слова**: магнітостатика, інтегральні рівняння, апроксимація, чисельне моделювання.*

Formulation of the problem

One of the important tasks of computational physics remains the development of universal algorithms for high-precision calculation and optimization of magnetostatic systems (MS) for a nonlinear environment, complex three-dimensional geometry of the magnetic system, as well as strong primary fields that can lead to saturation of the magneto conductor.

The first method of such calculation is MS method of finite differences (FDM). Unfortunately, this method is effective only in the simplest areas of a special form.

A more universal and modern method of calculation of MS is the finite element method (FEM) [1—2], which is implemented in many industrial software packages. But this method has many significant drawbacks. The main drawback of this method is the need to approximately set zero boundary conditions of the first kind on some boundaries in order to limit the calculation domain D. Usually, the calculation domain D consists of the magnetic domain G and the non-magnetic medium P region, i.e. $D = G \cup P$. Therefore, the boundaries of the domain P in FEM has to be chosen approximately, which adds to the calculation error. In addition, in this method, it is necessary to calculate the field parameters in the entire region D, both in G and in R. By the way, this drawback is also characteristic of the FDM. The FEM and FDM methods belong to the class of differential methods. They directly approximate the differential equations and boundary conditions of the MS calculation problem.

The class of methods of integral equations [3—6] is devoid of the mentioned shortcomings. They approximate the integral equations describing the MS. In this case, the field parameters are calculated only in the region of the magnets G, and not in the entire region D. There is no need to specify the boundary conditions either. It should be noted that integral methods are less studied than differential methods. Therefore, the implementation of such methods has not yet received proper justification.

In particular, the implementation of iterative methods for nonlinear integral equations raises a number of open questions concerning the convergence of the method of iteration of such equations and the influence of the nature of the discretization on the iterative solution.

Analysis of recent research and publications

There are two different approaches to the construction of integral equations of magnetostatics [3—6]. The first, the method of secondary sources [3], leads to equations for fictitious (calculated) field sources, distributed only on the boundary of the magnet region. The second approach [4—5], more modern and physical, leads to integral equations for spatially distributed field sources in the en-

tire area of magnets, not only at the boundary. In this case, the field sources are physical field vectors or their linear combinations, which is an advantage of this method. Most often, equations are used for the vector of magnetization, intensity or induction of the magnetic field or their linear combinations. This method of calculation is often called the method of spatial integral equations (SIE) [5].

At the first stage of SIE development, due to the insufficient speed of computer systems, the most urgent problem was the effective formation of a system of nonlinear equations (NLE) for a given discretization. Then, the solution of such NLE was reduced to the solution of the system of linear algebraic equations (LAE) by linearizing the NLE by taking into account the nonlinear characteristics of the environment. At the same time, the number of discretization elements was relatively small, their shape varied slightly, so the LAEs for such magnetic systems were well determined. The computing power of modern computers allows to dramatically increasing the number of MC discretization elements. However, in some cases it is difficult to provide a satisfactory iterative solution of NLE for such tasks, although theoretically all the convergence conditions of the iteration method for SIE are fulfilled.

Formulation of the purpose of the research

So, at the current stage of SIE development, one of the most urgent problems is the solution of NLE and LAE due to the large number of discretization elements and their significant variation in sizes and shapes. In addition, another circumstance complicating the situation is that the LAEs for the considered methods are not symmetric and not sign-defined. This leads to the fact that sometimes the resulting LAE for such MCs becomes ill-conditioned. The consequence of this may be a violation of the convergence conditions of the iterative process in case of unsuccessful discretization. It is difficult to apply direct methods that could provide a more accurate solution of the LAE due to regularization due to the large dimension of the LAE and the nonlinear nature of the MS. It should be emphasized that the stage of formation of NLE and LAE for SIE requires much more time than their iterative solution. Therefore, increasing the time of the NLE solution due to more complex but effective algorithms will not lead to a noticeable increase in the total time of MS calculation, however, it can decisively affect the quality of the calculation of the parameters of the magnetic system.

Presenting main material

The article proposes a block-by-block algorithm for calculating magnetostatic fields using the SIE method, which allows combining the advantages of direct and iterative methods of solving LAE for MS. Next, the spatial integral equation of the field with respect to the calculated vector is considered:

$$\overline{U} = \overline{B} / \mu_0 + \overline{H} = 2\overline{H} + \overline{M},$$

where $\overline{B}$ — induction vector, $\overline{H}$ — field strength vector, $\overline{M}$ — magnetization vector [4].

In turn, this ratio can be written as, $\overline{U} = 2 \cdot \overline{H}_0 + 2 \cdot \overline{H}_M + \overline{M}$. Here $\overline{H}_M$ — field strength created by the magnetization of the medium, a $\overline{H}_0$ — the intensity of the external field, which is created by the primary sources of the field — stationary current sources that do not change in time, and permanent magnets, that is, ferromagnetism with a predetermined and known magnetization vector that is practically independent of the external field. In this case, the modified integral equation is valid [4]

$$\overline{U}(x) = 2 \cdot \overline{H}_0(x) + \Pi\overline{M}, \qquad (1)$$

where $\Pi\overline{M} = 2 \cdot [\int_G K(x,y) \cdot \overline{M}(y) \cdot dv_y - N_\sigma \cdot \overline{M}] + \overline{M}$.

Here $K(x,y)$ — symmetric tensor of the second rank with components $K_{ij} = (\alpha_i \alpha_j - \delta_{ij})/R^m$; where $m = 2,3$ — dimension of space, α_{ii} — guiding cosines of the radius vector R; δ_{ij} — Kronecker symbol; changes in x,y are points in space, the integration is carried out only over the area of magnets G. For the case of a piecewise constant and isotropic medium, the parameter N_σ is zero [4]. We will consider only the isotropic medium used for the vast majority of MCs.

Next, only the calculation of the class of two-dimensional systems will be considered for simplification. The case of axisymmetric and three-dimensional systems is considered similarly to the case of two-dimensional systems. Equation (1) should be supplemented by the material magnetization curve

$B = B(H)$, with the help of which you can find the scalar nonlinear dependence $M = M(U)$ of the length of the vector M on the length of the vector U and connect these two quantities in one nonlinear integral equation.

In these conditions, it is necessary to find the distribution of the calculated vector U, and then with the help of the magnetization curve, the distribution of the magnetization vector M can be obtained. The field strength at any point can be obtained using integration over known field sources.

To obtain a numerical solution, the domain of magnets G is divided into a set of elementary domains $G = UGi$ in such a way that $mes(G_i \cap G_j) = 0$ for $i \neq j$. Within the elementary region, the properties of the environment will be considered constant. Therefore, in each individual element G_i the magnetization is constant, and thus, equation (1) is approximated using a piecewise constant approximation of the vector M.

Next, we turn to the discrete analog of equation (1). As a result, we will get a nonlinear vector equation relative to the values of the calculated vector $\overline{U}(x)$, which is considered in the centers of gravit x_i elements

$$\overline{U}(x_i) = 2\overline{H}_0(x_i) + 2\sum_{j=1}^{N} A_{ij} \cdot \overline{M}(x_j) + \overline{M}(x_i). \quad (2)$$

Here A_{ij} — tensor of the second rank. It is obtained by integration over the domain G_j of the function $K(x,y)$ and looks like $A_{ij} = \begin{pmatrix} T_{xx}, T_{xy} \\ T_{yx}, T_{yy} \end{pmatrix}$. The components of the tensor determine the components of the field strength at point x_i from the discretization element with number j.

For a piecewise constant medium, equation (2) is significantly simplified. It is equivalent to the equation

$$\overline{M} = 2\overline{H}_0 + (E + 2A - 2D)\overline{M} = 2\overline{H}_0 + S \cdot \overline{M}. \quad (3)$$

Matrix A consists of tensor cells A_{ij}, E — unit matrix. D cellular diagonal matrix in which 2×2 cells are located on the main diagonal. On the diagonal of the i-th cells are written with the same value $1/\chi_i$, and the other two elements of the cell are equal to zero. Here χ_i — value of the magnetic susceptibility in the i-th element..

The dimensionality of all matrices A, D and E is equal to $2N$, where N — number of discretization elements. This is true for two-dimensional and axisymmetric magnetic systems. For three-dimensional systems, the considerations are similar, only the cell sizes will be 3×3, not 2×2.

We immediately note, and this is important, that the matrix D is not constant, but depends on the vector M. Therefore, the iterative process for equation (3) must be modified in the following way

$$\overline{M}_{n+1} = 2\overline{H}_0 + S_n \cdot \overline{M}_n, \quad (4)$$

where $S_n = E+2(A - D_n)$.

In equation (4), there are actually two iterative processes: the main one with respect to the vector M_n, and the additional one with respect to the diagonal matrix D_n, which characterizes the properties of the ferromagnetic material. The size of the diagonal elements is recalculated according to the magnetization curve of the material after each iteration. The variable matrix D_n turns the iterative process (4) into a non-stationary one, which significantly complicates theoretical studies of the convergence of such a process. However, these processes are not equivalent: the main difficulty is the iterative process with respect to the M_n vector. This iterative process is a simple iteration method and should theoretically converge [4] for any MS and types of materials according to the theorem on the norm of the integral operator of equation (1), because the norm of this operator is theoretically less than unity.

But, as evidenced by the results of computational experiments, the module of the maximum eigenvalue of the matrix A can exceed unity as a result of the variation in the size and shape of the elements due to the instability of the computational process. Then sometimes the iterative process (4) will diverge. Another unpleasant circumstance is that the modulus of the minimum eigenvalue of the S_n matrix can be close to zero. It is known that if some eigenvalues of the matrix are equal to zero,

then such LAE has a non-unique solution. In practice, this can lead to "parasitic" solutions. Similar solutions for the discretized equation (4), which are caused by purely computational problems, will be called physically imprecise. This is caused by the fact that the eigenvalues of an unsymmetrical matrix are sometimes sensitive to rounding errors. In such cases, it is characteristic that the long iterative process (4) runs out (after 200—300 or more iterations). So, there can be two reasons for this:

1) maximum modulus of the number of the S_n matrix will be greater than the modulus of unity: $max|\lambda(S_n)| = 1+\delta_n$ for the iterative process (4), where $\delta_n > 0$;

2) minimum modulus number of the matrix $min|\lambda(S_n)|<\delta_n$.

In both cases, δ_n is a small number, on the order of 10^{-6}—10^{-8}.

A similar paradoxical situation, when it is not possible to obtain a more accurate solution for a greater number of iterations, sometimes occurs in the practice of MS calculations. To get out of this situation, the following is suggested.

If the number of elements is small, then the vector discretized equation (3) can be solved by direct methods, provided that the matrix D_n is already known. It is reduced to a system with respect to the magnetization vector M

$$R_n \cdot \overline{M} = (A - D_n) \cdot \overline{M}_n = -\overline{H}_0, \tag{5}$$

where $R_n = A - D_n$. In this case, no restrictions are imposed on the modulus of the maximum eigenvalue of the matrix R_n.

This method can be generalized. An approach that combines the advantages of direct and iterative methods for solving SIE can be called the method of block iterations (MBI). The area of magnets is divided into a small number of blocks m (where $m < 20$), which are united geometrically and are in approximately the same magnetic conditions, however, the latter is insignificant. We will perform (4) several general iterations over the entire system for an approximate determination of the matrix D_n. Numerical experiments show that only 5—10 iterations are enough for this. The matrix D_n is determined especially quickly for a linear medium.

Then we will perform block iterations. In this case, each block can be considered as a separate magnetic system and calculated according to (4) by iterative methods. The influence of other blocks can be considered as an external field. After the completion of the iteration process for the given block, we adjust the elements of the matrix D_n (but only for this block) and move on to the next block. Thus, after performing iterations on all blocks, we obtain a solution for the magnetic system — the distribution of the magnetization vector on all elements. The process ends when the relative deviation of the magnetic induction vectors for adjacent iterations throughout the system is less than the specified value. That is, $\left\|\overline{M}_{n+1} - \overline{M}_n\right\| / \left\|\overline{M}_{n+1}\right\| < \varepsilon$. It is important here that the theorem on the convergence of the sequential iteration method [4] is valid for each block, and therefore the block-by-block iteration process will converge.

As shown by numerical experiments, the maximum modulus number of the matrix $S_n(B_j)$ for the iterative process (4) $max|S_n(B_j)| < 1$ in all blocks. Moreover, all conditioning numbers for matrices $S_n(B_j)$, are, as a rule, smaller than the conditioning number of the entire matrix $S_n(G)$. The disadvantage of this method is that it takes longer to solve the NLE.

The algorithm of the block iteration method can be described as follows:

1. k (for example, k = 5—10) iterations over the entire domain G according to the iterative process (4). After that, the matrix D_n and the magnetization vector M_n are approximately determined.

2. In turn, in each block B_j, where $j = 1, m$, the iterative process (4) is performed, keeping unchanged the values of magnetization vectors for elements belonging to other blocks. The influence of other blocks is taken into account as an external field. As a variant of the algorithm, the magnetization vector can be obtained by the direct method (the Gaussian method with the selection of the main element by column), but by solving system (5) instead of (4). In this case, if necessary, Tikhonov regularization can be applied.

3. D_n matrix is adjusted for the discretization elements belonging to the given block on the magnetization curve of the material.

4. The transition to the next block is performed.

5. Steps 2—4 are repeated for those SIEs, until the relative error of the magnetization vector throughout the system becomes less than the specified one.

It is especially important that at stage 2 of the algorithm, direct and not iterative methods can be applied when solving the discretized equation only for a given block, or iterative and direct methods can be arbitrarily combined for different blocks. After all, the number of elements in the block is significantly less than in the entire system. In this case, you cannot worry about the fact that the module of the maximum eigenvalue of the system will be greater than one.

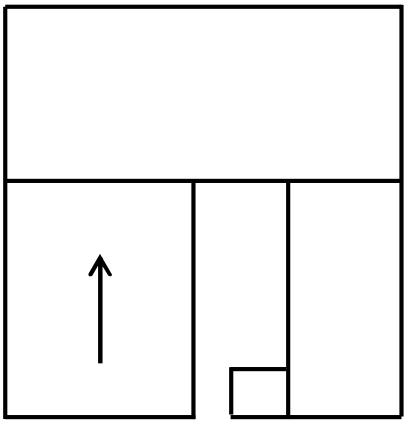

Fig. 1. Calculated magnetic system

For example, consider the calculation of the magnetic field of a typical flat two-dimensional MS, the sketch of which is shown in Fig. 1. The geometry of the MS is specially simplified to illustrate the proposed method. To obtain a strong magnetic field, a specially designed magnetic conductor is used, which must concentrate the entire magnetic flux in a small gap below. The magnetic wire of this MS consists of three ferromagnetic rectangles of significantly different sizes: the first has an area of 2, the second — 0.5, and the third — 0.04 area units. The material of the magnet wire is electrical steel, the calculation was carried out with a real magnetization curve. As the primary source of the field, a solid-magnet material (magnet) with a vertical magnetization of 0.7 T was chosen (element with an arrow, lower left). The most important for the calculation is the field exactly in the smallest rectangle of the MC. Therefore, it is theoretically advisable to break it down as small as possible. But if the discretization of two large rectangles will be significantly different from the discretization in a small one, then this will lead to a bad conditioning of the LAE. The opposite output — discretization into approximately the same elements — will lead to a large number of them and significant program runtime. Therefore, the proposed method of block iterations is a logical solution to such a problem.

In the tabl. 1—4 show the data for evaluating the convergence of the iterative processes of calculating the magnetic field of a given MS according to the algorithm of the block-by-block iteration method. Two modifications of the iterative process were considered: with the matrices $S_n = E+2(A - D_n)$ and $R_n = A - D_n$. For the analysis in the process of computational experiments, the QR method was used to calculate the values of the maximum and minimum values of the modulus of the eigenvalues of the matrices of the corresponding blocks and the modulus of the conditioning number of the LAE corresponding to certain discretizations. The numbering of the blocks in the tables corresponds to the number specified when describing their sizes, and the parameters for the entire system are marked as a block with the number 0. The second column of the tables shows the discretization parameters. The first number indicates the number of divisions of the vertical side, the second - the number of divisions of the horizontal side. Note that the magnet does not need to be discretized, so it adds only one element. In the tabl. 1 and 2 show the results for the case of "satisfactory" discretization, and Tabl. 3 and 4, the discretization was emphatically irrational, because the elements had an elongated shape, which, as practice shows, leads to poor conditioning of the LAE and, as a result, to poor convergence of the iterative process. But in real complex systems, it is difficult to propose an ideal discretization, so it was important to investigate the effect of "bad" discretization on the iterative process.

Analyzing the results of the numerical experiment, it can be seen that for all cases, the block iteration method has better iterative properties than the general iterative process (block conditioning number 0). Second, the R_n matrix has better iterative properties than the S_n, matrix for almost all discretizations. And in the case of "bad" discretization (Tabl. 4), the iterative properties of the matrix R_n are better than in the three previous cases.

Table 1. Matrix parameters S_n

Block number	The number of discretization elements	Condition number	Module max. own number	Module min. own number
0	67	236.558	0.97762	0.004133
1	4×4	154.421	0.97652	0.006324
2	5×5	34.6258	0.97205	0.028073
3	5×5	97.0014	0.96398	0.009938

Table 2. Matrix parameters R_n

Block number	The number of discretization elements	Condition number	Module max. own number	Module min. own number
0	67	78.6684	0.98881	0.0125693
1	4×4	76.4524	0.98822	0.0129265
2	5×5	66.7421	0.98602	0.0147737
3	5×5	53.1098	0.98198	0.0184898

Table 3. Matrix parameters S_n

Block number	The number of discretization elements	Condition number	Module max. own number	Module min. own number
0	84	343.5336	0.97437	0.002836
1	4×7	314.4429	0.96789	0.003078
2	8×5	245.8907	0.96708	0.003078
3	3×5	12.44069	0.91058	0.003933

Table 4. Matrix parameters R_n

Block number	The number of discretization elements	Condition number	Module max. own number	Module min. own number
0	84	73.0886	0.98718	0.0135067
1	4×7	57.5418	0.98395	0.0170997
2	8×5	57.3264	0.98354	0.0171569
3	3×5	21.1625	0.95529	0.0451406

Conclusions

1. The method of block iterations has a wider field of application than the iteration method, due to the fact that it allows you to effectively combine the advantages of direct and iterative methods for solving the nonlinear discretized equation of magnetostatics.

2. The method of block-by-block iterations allows obtaining a reliable solution even in cases where the method of iterations over the entire system (classical iteration method) diverges or leads to a physically incorrect solution.

3. In the case of using direct methods of intra-block iteration, it is possible to effectively apply the regularization of part of the Tikhonov equation.

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УЗАГАЛЬНЕНИЙ СПОСІБ ВИРІШЕННЯ РІВНЯННЯ МАГНІТОСТАТИКИ

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Однією з важливих задач обчислювальної фізики залишається розробка універсальних алгоритмів високоточного розрахунку і оптимізації магнітостатичних систем (МС) для нелінійного середовища, складної трьохвимірної геометрії магнітної системи, а також сильних первинних полів, що можуть призводити до насичення магнітопроводу.

Найбільш універсальним та сучасним методом розрахунку МС є метод скінчених елементів (МСЕ), що реалізований в багатьох промислових пакетах програм. Але цей метод має багато істотних вад. Мабуть, найголовнішим недоліком цього методу є потреба штучно задавати приблизно нульові крайові умови першого роду на деяких границях, щоб обмежити область розрахунку. Ця область складається з області магнетиків G та області немагнітного середовища *P*. Отже, границі області *P* доводиться вибирати приблизно, що додає похибку розрахунку. Крім того, в цьому методі потрібно розраховувати параметри поля у всій області *D*, як в G так і в *P*. До речі, цей недолік властивий також і методу скінчених різниць. Методи МСЕ та МСР належать до класу диференціальних методів. Вони безпосередньо апроксимують диференціальні рівняння та крайові умови задачі розрахунку МС.

Цих недоліків позбавлений клас методів інтегральних рівнянь. Вони апроксимують інтегральні рівняння, що описують МС. Розрахунок параметрів поля в проводиться тільки в області магнетиків *G*, а не всій області *D*. Слід помітити, що інтегральні методи менш досліджені, ніж диференціальні методи. Тому реалізація таких методів ще не отримала належного обгрунтування в повній мірі на даний час.

Зокрема, реалізація ітераційних методів для таких рівнянь ставить цілий ряд відкритих питань, що стосуються збіжності методу ітерації для нелінійних інтегральних рівнянь та впливу характеру дискретизації на ітераційне рішення.

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