МАТЕМАТИЧНІ МЕТОДИ В СУСПІЛЬНИХ І ГУМАНІТАРНИХ НАУКАХ

DOI: 10.31319/2519-8106.2(47)2022.268416
UDK 330.101:542.075.8
M. Boychuk, Ph. D., Associate Professor
O. Vinnychuk, Ph. D., Associate Professor, o.vinnychyk@chnu.edu.ua
L. Skrashchuk, Ph.D., Assistant, l.skrashchuk@chnu.edu.ua
Yuriy Fedkovych Chernivtsi National University, Chernivtsi

STOCHASTIC MODELLING OF SINGLE-PRODUCT GROWTH MACROECONOMICS WITH ENDOGENOUS SCIENTIFIC AND TECHNOLOGICAL PROGRESS UNDER INVESTMENT LAG

A stochastic model of a single-product growth economy with endogenous scientific and technological progress under investment lag with Wiener and Poisson processes is proposed. In the stochastic economic-mathematical model it is taken into account that the final output is used for consumption, for investment in the expansion of fixed assets, for improving production, taking into account the cost-effectiveness of "science", for taxation, for government expenditures, for the balance and for the elimination of environmental pollution. This model takes into account the impact of research on economic production in the system itself, i.e. this impact is an endogenous variable. Stochastic sufficient conditions of optimality are used to study the stochastic optimal ecological-economic model.

An algorithm for calculating the optimal process when choosing the necessary economic boundary mode at the initial stage, as well as an algorithm for calculating the optimal process when choosing the economic main mode when choosing at the initial stage, is constructed.

***Keywords**: endogenous scientific and technological progress, boundary process, main process, control switching moment, optimal process.*

Запропонована стохастична модель однопродуктової економіки зростання з ендогенним науково-технічним прогресом при інвестиційному запізненні з вінерівськими та пуассонівськими процесами. В стохастичній економіко-математичній моделі враховано, що кінцевий випуск продукції використовується на споживання, на капіталовкладення в розширення основних фондів, на покращення виробництва з урахуванням ефективності затрат «на науку», на оподаткування, на урядові витрати, на сальдо та на ліквідацію забруднення навколишнього середовища. Ця модель враховує вплив наукових досліджень на економічне виробництво у самій системі, тобто цей вплив є ендогенною змінною. Для дослідження стохастичної оптимальної еколого-економічної моделі використано стохастичні достатні умови оптимальності.

При дослідженні стохастичної економіко-математичної моделі побудовано алгоритм розрахунку оптимального процесу при виборі необхідного економічного крайового режиму на початковій стадії, а також алгоритм розрахунку оптимального процесу при виборі економічного магістрального режиму при виборі на початковій стадії.

***Ключові слова**: ендогенний науково-технічний прогрес, крайовий процес, магістральний процес, момент перемикання керування, оптимальний процес.*

Problem's formulation

The impact of scientific and technological progress on the nature of growth in the economic system is manifested in various forms, in particular when considering non-autonomous, that is, time-varying macro-production functions. Failure to take into account some economic indicators in economic and mathematical models, uncertainty and inaccuracy of input information leads to stochastic modelling in the study of economic systems and processes. This article considers a stochastic economic-mathematical model with the use of Wiener and Poisson processes, where the impact of scientific research on production is programmed in the system itself, that is, in the model with internal (endogenous) consideration of scientific and technological progress. Therefore, it is relevant, both in theoretical and practical terms, to study the impact of scientific research on production in the endogenous system itself and with an investment lag.

Analysis of recent research and publications

The paper [1] proposes a deterministic model of single-product macroeconomics of growth with endogenous accounting of scientific and technological progress. However, it does not include consumption. The model was studied using the necessary conditions of optimality (Pontryagin's principle). And in [2], a study was conducted using sufficient optimality conditions for deterministic single-product macroeconomics of growth, taking into account consumption, labor resources (labor force) and endogenous scientific and technological progress. In [3], the use of Wiener and Poisson processes in stochastic modelling of economic dynamical systems is economically justified.

Formulation of study purpose

The purpose of the work is construction of stochastic model of single-product macroeconomics of growth taking into account endogenous scientific and technological progress and investment lag using Wiener and Poisson processes and conducting its research.

Presenting main material

First, we formalize a deterministic model of a single-product growth macroeconomy with endogenous technological progress and investment lags. We describe the assumptions for the construction of a deterministic economic-mathematical model.

Assumption 1. Labor resources (labor force) L at a point in time $t-\tau$ ($t \ge t_0$, $t_0 \ge 0$ — the beginning of time, $\tau \ge 0$ — investment lag) are subject to an exponential law $L(t-\tau) = L_0 e^{n(t-t_0-\tau)}$ with known initial state of the labor force $L_0 > 0$ and the growth rate of the labor force $n > 0$ and is a solution to the initial task $\dot{L}(t) = nL(t)$, $L(t_0) = L_0$, where $\dot{L}(t) = \frac{dL(t)}{dt}$ — is the increment of the labor force.

Assumption 2. Let gross (intermediate) output X is decomposed into final output Y та and production consumption W [4]: $X(t-\tau) = Y(t-\tau) + W(t-\tau)$, $t_0 \ge 0$; production consumption W is directly proportional to gross output X: $W(t-\tau) = aX(t-\tau)$, $t \ge t_0$, $a \in (0;1)$. Then the final product Y is determined by $Y(t-\tau) = (1-a)X(t-\tau)$, $t \ge t_0$.

Assumption 3. Final output Y is decomposed into non-productive consumption, general investments I, government expenditure U_r, taxation O_p, pollution elimination Z_a and balance (exports minus imports) S_a:

$$Y(t-\tau) = C(t-\tau) + I(t-\tau) + U_r(t-\tau) + O_p(t-\tau) + Z_a(t-\tau) + S_a(t-\tau),\ t \ge t_0.$$

Let the total costs of government expenditures U_r, taxation O_p, pollution elimination Z_a and balance S_a are directly proportional to the final output Y:

$$U_r(t-\tau) + O_p(t-\tau) + Z_a(t-\tau) + S_a(t-\tau) = bY(t-\tau),\ t \ge t_0.$$

Then total consumption C and total investment I are:

$$C(t-\tau) + I(t-\tau) = (1-b)Y(t-\tau) = (1-a)(1-b)X(t-\tau),\ t \ge t_0.$$

Assumption 4. Consumption C is directly proportional to the share of final output $(1-b)Y$ with the proportionality coefficient $s \in (0;1)$ depending on $t-\tau$ (consumption rate):

$$C(t-\tau) = s(t-\tau)(1-b)Y(t-\tau),\ t \ge t_0.$$

Then the total investment I is determined by

$$I(t-\tau) = (1-b)\left[1 - s(t-\tau)\right]Y(t-\tau) = \left[1 - s(t-\tau)\right](1-b)(1-a)X(t-\tau),\ t \ge t_0.$$

Assumption 5. Total investment is divided into investment in capital accumulation I_{ac} and investment in scientific and technological progress ("science") I_{sc}

$$I(t-\tau)=I_{ac}(t-\tau)+I_{sc}(t-\tau),\ t\ge t_0.$$

Assumption 6. Let investment in capital accumulation I_{ac} are proportional to the share of final output $(1-b)\left[1-s(t-\tau)\right]Y(t-\tau)$:

$$I_{ac}(t-\tau)=(1-b)\left[1-s(t-\tau)\right]u(t-\tau)Y(t-\tau)=$$
$$=(1-a)(1-b)\left[1-s(t-\tau)\right]u(t-\tau)X(t-\tau),\ u\in[0,1],\ t\ge t_0.$$

Then the investments "in science" are:

$$I_{sc}(t-\tau)=(1-b)\left[1-s(t-\tau)\right]\left[1-u(t-\tau)\right]Y(t-\tau)=$$
$$=(1-a)(1-b)\left[1-s(t-\tau)\right]\left[1-u(t-\tau)\right]X(t-\tau),\ \ t\ge t_0.$$

Assumption 7. Gross output X is a macro production function $F(K,L)$ with endogenous technological progress

$$X(t-\tau)=A\big(Q(t-\tau)\big)F\big(K(t-\tau),L(t-\tau)\big),\ t\ge t_0,$$

where $F(K,L)$ — is a macro production function with properties [4]: $F(0,0)=F(0,L)=F(K,0)=0$, $F(K\ge 0,L\ge 0)\ge 0$ — twice continuously differentiated, monotonically increasing by K and L $(F'_K>0,F'_L>0)$, concave by K and L $\left(F''_{K^2}<0,F''_{L^2}<0\right)$; $\lim\limits_{K\to\infty}F(K,L)=\lim\limits_{L\to\infty}F(K,L)=\infty$, $\lim\limits_{K\to 0}F'_K=\lim\limits_{L\to 0}F'_L=\infty$, $\lim\limits_{K\to\infty}F'_K=\lim\limits_{L\to\infty}F'_L=0$; homogeneous degree $v\in(0,2)$. Function $F(K,L)=L^vF(K/L,1)=L^vf(k)$ with specific macro-production function: twice continuously differentiable, monotonically increasing, concave and $f(0)=0$; where $k=K/L$ — capital intensity (specific capital); $A(Q\ge 0)>0$ — the effectiveness of the impact of scientific and technological progress on production; Q — the volume of investments in scientific and technological progress.

Assumption 8. The dynamics of capital flows is described by a differential model — investment in capital accumulation equals the sum of capital gains (net investment) and depreciation (μK):

$$\dot K(t)+\mu K(t)=I_{ac}(t-\tau)=(1-a)(1-b)\left[1-s(t-\tau)\right]\times$$
$$\times u(t-\tau)A\big(Q(t-\tau)\big)F\big(K(t-\tau),L(t-\tau)\big),\ t\ge t_0.$$

Assumption 9. Increase in investment in scientific and technological progress is equal to investment "in science"

$$\dot Q(t)=I_{sc}(t-\tau)=(1-a)(1-b)\left[1-s(t-\tau)\right]\left[1-u(t-\tau)\right]\times$$
$$\times A\big(Q(t-\tau)\big)F\big(K(t-\tau),L(t-\tau)\big),\ t\ge t_0.$$

From assumptions 1—9 we obtain the mathematical model

$$\begin{cases}\dot K(t)=-\mu K(t)+(1-a)(1-b)\left[1-s(t-\tau)\right]\times\\ \times u(t-\tau)A\big(Q(t-\tau)\big)F\big(K(t-\tau),L(t-\tau)\big),\\ \dot Q(t)=(1-a)(1-b)\left[1-s(t-\tau)\right]\left[1-u(t-\tau)\right]\times\\ \times A\big(Q(t-\tau)\big)F\big(K(t-\tau),L(t-\tau)\big),\ 0\le u(t-\tau)\le 1,\ t\ge t_0,\\ K(y)=K_0(y),\ Q(y)=Q_0(y),\ y\in[t_0-\tau,t_0],K(T)=K_T.\end{cases}\qquad(1)$$

with a constraint on the rate of capital accumulation and on the final state of capital of the system and under given initial conditions with prehistory.

Let us write model (1) in specific indicators: $k = K / L$ — capital intensity (specific capital), $q = Q / L$ — specific volume of investments in scientific and technological progress, $k_0 = K_0 / L_0$, $q_0 = Q_0 / L_0$, $k_T = K_T / L(T)$.

From (1) and the equations

$$L(t-\tau)/L(t) = e^{-n\tau},\ \dot{k} = (K/L)' = \dot{K}/L - K/L \cdot \dot{L}/L = \dot{K}/L - nk,\ \dot{q} = \dot{Q}/L - nq$$

we obtain the economic-mathematical model in specific indicators

$$\begin{cases} \dot{k}(t) = -(\mu+n)k(t) + (1-a)(1-b)\left[1-s(t-\tau)\right]u(t-\tau)A\big(Q(t-\tau)\big)\times \\ \times L_0^{v-1} f\big(k(t-\tau)\big)e^{(v-1)n(t-t_0-\tau)-vn\tau}, \\ \dot{q}(t) = -nq(t) + (1-a)(1-b)\left[1-s(t-\tau)\right]\left[1-u(t-\tau)\right]A\big(Q(t-\tau)\big)\times \\ \times L_0^{v-1} f\big(k(t-\tau)\big)e^{(v-1)n(t-t_0-\tau)-vn\tau}, \\ 0 \le u(t-\tau) \le 1,\ Q(t-\tau) = L_0 q(t-\tau)e^{n(t-t_0-\tau)},\ t \ge t_0,\ k(T) \ge k_T, \\ k(y) = k_0(y),\ q(y) = q_0(y),\ y \in [t_0-\tau, t_0]. \end{cases} \tag{2}$$

Now, based on the deterministic economic-mathematical model (2), we formalize the stochastic model. To do this, we use the results of [3], according to which the use of Wiener and Poisson processes in stochastic modelling of economic systems is economically justified.

Stochastic economic and mathematical model. Let $\{\Omega, \Im, P\}$ is a probability space with σ — algebra $\{\Im_t, t \in [t_0, T]\} \subset \sigma$, with a set of elementary events Ω and to the extent (probability) P; $\xi_i(t) \equiv \xi_i(t,\omega) \in \mathbb{R}$ ($\mathbb{R}$ — set of real numbers) $\in \Im_t$ — measurable standard Wiener process with zero mathematical expectation $M\xi_i(t) = 0$ and unit variance $M\xi_i^2(t) = 1$, $\omega \in \Omega$, $t \ge t_0$, $i = 1,2$, $\eta(t) \equiv \eta(t,\omega) \in \mathbb{R}$ is $\Im_t$ — measurable Poisson process with mathematical expectation $M\eta_i(t) = \lambda_i(t-t_0)$, $\lambda_i \equiv const$, $i = 1,2$, $\omega \in \Omega$, $t \ge t_0$, where, Wiener processes $\xi_i(t)$ and Poisson processes $\eta_i(t)$, $i = 1,2$, $t \ge t_0$ re independent random processes [5].

On the probability space $\{\Omega, \Im, P\}$ are given random processes of specific capital $k(t) \equiv k(t,\omega)$ and specific volume of investments in scientific and technological progress $q(t) \equiv q(t,\omega)$, $\omega \in \Omega$, $t \ge t_0$ and satisfying

- equation of dynamics of movement of specific capital and specific volume of investments in scientific and technological progress in the form of Ito [5,6]

$$\begin{cases} \dot{k}(t) = -(\mu+n)k(t) + (1-a)(1-b)\left[1-s(t-\tau)\right]u(t-\tau)A\big(Q(t-\tau)\big)\times \\ \times f\big(k(t-\tau)\big)e^{(v-1)n(t-t_0-\tau)-vn\tau} + \alpha_1\dot{\xi}_1(t) + \beta_1\dot{\eta}_1(t), \\ \dot{q}(t) = -nq(t) + (1-a)(1-b)\left[1-s(t-\tau)\right]\left[1-u(t-\tau)\right]A\big(Q(t-\tau)\big)\times \\ \times f\big(k(t-\tau)\big)e^{(v-1)n(t-t_0-\tau)-vn\tau} + \alpha_2\dot{\xi}_2(t) + \beta_2\dot{\eta}_2(t), \\ Q(t-\tau) = L_0 q(t-\tau)e^{n(t-t_0-\tau)},\ t \ge t_0, \end{cases} \tag{3}$$

- initial conditions with background

$$k(y) \equiv k_0(y),\ q(y) \equiv q_0(y),\ k_0 \in \Im_{t_0},\ q_0 \in \Im_{t_0},\ y \in [t_0-\tau, t_0], \tag{4}$$

- restrictions on the final state of the specific capital

$$k(T) = k_T. \tag{5}$$

Here the derivatives of the Wiener processes $\dot{\xi}_i(t)$, $i=1,2$ and Poisson processes $\dot{\eta}_i(t)$, $i=1,2$ should be understood as generalized, i.e. derivatives of functionals [7]; functions k_0 and q_0 — are piecewise continuous on $[t_0-\tau,t_0]$.

Restrictions are imposed on the rate of capital accumulation

$$0\le u(t-\tau)\le 1,\ t\ge t_0. \tag{6}$$

Let $D_\varepsilon=\{(k,q)\in\mathbb{R}^2 \,|\, k_T-\varepsilon\le k(t)\le k_T+\varepsilon,\ q(t)\ge 0,\ t\ge t_0\}$ ($\varepsilon>0$ — a sufficiently small given number) — ε-closed envelope of the final state of the system k_T. Let $T_u(k,q)$ denote the moment of the first reaching of the set D_ε by the system (3), starting the motion of the point (k_0,q_0) at $y\in[t_0-\tau,t_0]$.

This stochastic optimal performance problem consists in choosing such a control $u_{ОП}(t-\tau)$, $t\ge t_0$, at which the average time of the first reaching of the set D_ε by the moving point is minimal

$$T_{u_{ОП}}(k,q)=\min_{0\le u\le 1} MT_u(k,q). \tag{7}$$

Let us study the stochastic optimal performance of (3)—(4), (7) using stochastic sufficient conditions of optimality [8].

Control. Write the Bellman equation with boundary condition

$$\begin{cases}\inf\limits_u R\big(t,k(t),q(t),s(t-\tau),\varphi_1(t),\varphi_2(t),u(t-\tau),V\big)\equiv\inf\limits_u\Big\{\partial V/\partial t+\\ +\partial V/\partial k\Big[-(\mu+n)k+(1-a)(1-b)L_0^{v-1}(1-s)uA(\varphi_1)f(\varphi_2)e^{(v-1)n(t-t_0-\tau)-vn\tau}\Big]+\\ +\partial V/\partial q\Big[-nq+(1-a)(1-b)L^{\gamma-1}(1-s)(1-u)A(\varphi_1)f(\varphi_2)e^{(v-1)n(t-t_0-\tau)-vn\tau}\Big]+\\ +0,5\alpha_1^2\,\partial^2V/\partial k^2+0,5\alpha_2^2\,\partial^2V/\partial q^2+\lambda_1\big[V(t,k+\beta_1,q,\varphi_1,\varphi_2)-V(t,k,q,\varphi_1,\varphi_2)\big]+\\ +\lambda_2\big[V(t,k,q+\beta_2,\varphi_1,\varphi_2)-V(t,k,q,\varphi_1,\varphi_2)\big]+1\Big\}=0,\\ \varphi_1(t)\equiv Q(t-\tau)=L_0q(t-\tau)e^{n(t-t_0-\tau)},\ \ \varphi_2(t)\equiv k(t-\tau),\ \ t\ge t_0,\\ V\big(t,k_T,q(T),\varphi_1(T),\varphi(T)\big)=0,\ \ \forall\varphi_1(T)\ge 0,\varphi_2(T)\ge 0,\end{cases} \tag{8}$$

where the desired function $V(t,k,q,\varphi_1(t),\varphi_2(t))$ — is continuously differentiable once by t and twice by k and q at all piecewise continuous on $t\ge t_0$ the functions φ_1 and φ_2, where φ_1 and φ_2 are the parameters in the Bellman equation (8). The unknown function V will be sought in the form

$$V(t,k,q,\varphi_1,\varphi_2)=k^2+q^2+l_1k+l_2q-k_T^2-q^2(T)-l_1k_T-l_2q(T), \tag{9}$$

where constant l_1 and l_2 are determined (selected).

Substitute (9) into the Bellman equation (8). The function R is linear by u, and therefore on the interval [0,1] by u the smallest value is obtained at $\varphi_1\equiv Q(t-\tau)$ and $\varphi_2\equiv k(t-\tau)$ and boundary controls by the rate of capital accumulation

$$u_b(t-\tau)=\begin{cases}1\ \text{at}\ \partial V/\partial k-\partial V/\partial q=2k+l_1-2q-l_2<0,\\ 0\ \text{at}\ \partial V/\partial k-\partial V/\partial q=2k+l_1-2q-l_2>0,\\ \text{any of } [0,1]\ \text{at}\ \partial V/\partial k-\partial V/\partial q=2k+l_1-2q-l_2=0, t\ge t_0,\end{cases} \tag{10}$$

where $\partial V / \partial k = 2k + l_1$ — characterizes the rate of efficiency of capital accumulation, $\partial V / \partial q = 2q + l_2$ — rate of efficiency of investments in scientific and technological progress (in science), accordingly.

Let's consider a special case, when $\Phi(k,q) = \partial V / \partial k - \partial V / \partial q = 2k - 2q + l_1 - l_2 = 0$ — is a special curve in phase space kOq. Under the special curve $\Phi(k,q) > 0$ marginal control by the rate of capital accumulation is $u_b(t-\tau) = 0$, and above — is the boundary control $u_b(t-\tau) = 1$, $t \ge t_0$. This special curve determines the control switching moments as well as the so-called main control, because the special line can be called a main line.

Let's define the main controls. On a special curve $\Phi(k,q) = 0$

$$k(t) - q(t) + 0{,}5(l_1 - l_2) = 0,\ t \ge t_0, \tag{11}$$

we write down the Bellman equation at $\varphi_1(t) \equiv Q(t-\tau)$ and $\varphi_2(t) \equiv k(t-\tau)$:

$$\begin{cases} -(\mu+n)\left[2k(t)+l_1\right]k(t) + n\left[2q(t)+l_2\right]q(t) + (1-a)(1-b)L_0^{n-1}\left[1-s(t-\tau)\right]\times \\ \times A(Q(t-\tau)) f(k(t-\tau)) e^{(v-1)n(t-t_0-\tau)-vn\tau}\left[2q(t)+l_2\right] + \\ +\alpha_1^2 + \alpha_2^2 + \lambda_1\left[(k(t)+\beta_1)^2 + l_1(k(t)+\beta_1) - k^2(t) - l_1 k(t)\right] + \\ +\lambda_2\left[(q(t)+\beta_2)^2 + l_2(q(t)+\beta_2) - q^2(t) - l_2 q(t)\right] + 1 = 0,\ k(t) \ge 0,\ q(t) \ge 0, \\ t \ge t_0,\ k(y) = Mk_0(y),\ q(y) = Mq_0(y),\ y \in [t_0 - \tau, t_0]. \end{cases} \tag{12}$$

The system of nonlinear algebraic equations (11)—(12) is obtained by determining the optimization quantities $k_{l_1 l_2}(t)$ and $q_{l_1 l_2}(t)$, $t \ge t_0$ and which can be solved by using the method of steps [9] and one of the numerical methods for solving a system of nonlinear equations [6,10] or one nonlinear equation obtained by substitution $q(t) = k(t) + 0{,}5(l_1 - l_2)$ into equation (12). If the system of equations (11)—(12) has no nonnegative solutions $k(t) \ge 0$ and $q(t) \ge 0$, $t \ge t_0$, then it is necessary to relax the conditions for the input information of the economic-mathematical model (3)—(4).

Let the vector of nonnegative solutions $(k_{l_1 l_2}, q_{l_1 l_2})$ of the system (11)—(12) exist. There can be at most two vectors of nonnegative solutions, since equation (12) is quadratic in k or q.

According to the found optimization values $k_{l_1 l_2}$ and $q_{l_1 l_2}$ (there are no more than two pairs of them) from the average dynamics of specific capital and specific volume of investments in scientific and technological progress of the system (1). Using the properties of Wiener and Poisson processes

$$M\dot{\xi}_i(t) = (M\xi_i(t))' = 0,\ M\dot{\eta}_i(t) = (M\eta_i(t))' = (\lambda_i(t-t_0))' = \lambda_i,$$

for the system (3) formally write down the system of mean dynamics (should be included in the stochastic optimality conditions)

$$\begin{cases} \dot{k}(t) = -(\mu+n)k(t) + (1-a)(1-b)\left[1-s(t-\tau)\right]u(t-\tau)A(Q(t-\tau))\times \\ \times f(k(t-\tau)) e^{(v-1)n(t-t_0-\tau)-vn\tau} + \lambda_1\beta_1, \\ \dot{q}(t) = -nq(t) + (1-a)(1-b)\left[1-s(t-\tau)\right]\left[1-u(t-\tau)\right]A(Q(t-\tau))\times \\ \times f(k(t-\tau)) e^{(v-1)n(t-t_0-\tau)-vn\tau} + \lambda_2\beta_2,\ t \ge t_0. \end{cases} \tag{13}$$

From the system (13) we obtain the main controls:

- from the average dynamics of the specific capital

$$u_m^1(t-\tau)=\left[\dot{k}_{l_1l_2}(t)+(\mu+n)k_{l_1l_2}(t)-\lambda_1\beta_1\right](1-a)^{-1}(1-b)^{-1}\times$$
$$\times\left[1-s(t-\tau)\right]^{-1}A^{-1}\left(Q_{l_1l_2}(t-\tau)\right)f^{-1}\left(k(t-\tau)\right)e^{(1-\nu)n(t-t_0-\tau)+\nu n\tau},\ t\in[t_0,T]; \quad (14)$$

- from the average dynamics of the specific volume of investments in scientific and technological progress

$$u_m^2(t-\tau)=1-\left[\dot{q}_{l_1l_2}(t)+nq_{l_1l_2}(t)-\lambda_2\beta_2\right](1-a)^{-1}(1-b)^{-1}\times$$
$$\times\left[1-s(t-\tau)\right]^{-1}A^{-1}\left(Q_{l_1l_2}(t-\tau)\right)f^{-1}\left(k_{l_1l_2}(t-\tau)\right)e^{(1-\nu)n(t-t_0-\tau)+\nu n\tau},\ t\geq t_0, \quad (15)$$

where $Q_{l_1l_2}(t-\tau)=L_0q_{l_1l_2}(t-\tau)e^{n(t-t_0-\tau)}$, $t\geq t_0$.

It should be noted that the choice constant l_1 and l_2 it is possible to achieve that the main control by the rate of capital accumulation $u_m^1\in[0,1]$, $u_m^2\in[0,1]$ and such pairs can be no more than two.

According to the found boundary controls $u_b(t-\tau)=0$, $u_b(t-\tau)=1$ piecewise continuous on $t\geq t_0$ the main controls $u_m=u_m^1$, $u_m=u_m^2$ the corresponding marginal trajectories k_b, q_b piecewise differentiable on $t\geq t_0$ the main trajectories k_m, q_m by the specific capital and the specific volume of investments in scientific and technological progress are defined from the corresponding stochastic initial problems with rehistories, and the averages are $k_b^{(c)}(t)=Mk_b(t)$, $q_b^{(c)}(t)=Mq_b(t)$, $k_m^{(c)}(t)=Mk_m(t)$ and $q_m^{(c)}(t)=Mq_m(t)$, $t\geq t_0$.

Thus, we have obtained two boundary processes $\{k_b(t),q_b(t),u_b(t-\tau),\ t\geq t_0\}$ at least two and no more than four main processes $\{k_m(t),q_m(t),u_m(t-\tau),\ t\geq t_0\}$, that is, we have obtained at least four and maximum six economic modes, including two boundary modes.

The economic system moves along the boundary trajectories $k_b(t)$ and $q_b(t)$ until it reaches a special straight curve $\Phi(k,q)=0$ in the phase plane kOq under the boundary control u_b, then moves along a special straight curve under the chosen main control until to the descent from it and further movement — along new boundary trajectories at the boundary control $u_b(t-\tau)=1$, $t\geq t_0$ until reaching the final state by specific capital k_T.

The algorithm for calculating optimal processes is presented depending on the location of the initial point (k_0,q_0): under, above and belonging to a special curve $\Phi(k,q)=0$.

Algorithm for calculating the optimal process when choosing an boundary mode at the initial stage

1. Let us choose the necessary stochastic and mean boundary conditions, and accordingly the boundary process $\Pi_b=\{k_b(t),q_b(t),u_b(t-\tau),t\geq t_0\}$: when the inequality is satisfied $\Phi(Mk_0(y),Mq_0(y))=2Mk_0(y)-2Mq_0(y)+(l_1-l_2)<0$, $y\in[t_0-\tau,t_0]$ (initial state (Mk_0,Mq_0) lies under the special straight curve $\Phi(k,q)=0$) with boundary control $u_b(t-\tau)=0$, $t\geq t_0$, and at inequality $\Phi(Mk_0(y),Mq_0(y))=2Mk_0(y)-2Mq_0(y)+(l_1-l_2)>0$, $y\in[t_0-\tau,t_0]$ (the initial state (Mk_0,Mq_0) lies above the special straight curve $\Phi(k,q)=0$) — with boundary control $u_b(t-\tau)=1$, $t\geq t_0$.

2. The economic system moves along boundary trajectories $k_b(t)$ and $q_b(t)$ with boundary control $u_b(t-\tau)$, $t\geq t_0$ until meeting with a special straight curve $\Phi(k,q)=0$. The moment of meeting ζ_1 is the moment of switching control and is defined by one of the numerical methods [6, 10, 11] from the nonlinear algebraic equation (equation of the special straight curve)

$$\Phi(k_b(t),q_b(t))=2Mk_b(t)-2Mq_b(t)+(l_1-l_2)=0,\ t\geq t_0.$$

After that, the economic system moves along the selected main trajectories $k_m(t)$ and $q_m(t)$, $t \ge \zeta_1$ under the selected main control $u_m(t-\tau)$, main process $\Pi_m = \{k_m(t), q_m(t), u_m(t-\tau),\ t \ge \zeta_1\}$ to the moment ζ_2 descent from the special straight curve $\Phi(k,q)=0$, $t \ge \zeta_1$.

3. The moment ζ_2 is the moment of control switching and is the greatest root (solution) of the equation $\Phi(k_m(t), q_m(t)) = 2Mk_m(t) - 2Mq_m(t) + (l_1 - l_2) = 0$, $t \ge \zeta_1$. The moment ζ_2 can be found by the search method.

4. Further movement of the economic system is carried out according to the new boundary regime (process) $\Pi_b^{(n)} = \{k_b^{(n)}(t), q_b^{(n)}(t), u_b(t-\tau)=1,\ t \ge \zeta_2\}$ to the moment T (the end of the research of economic process), which is the solution of the nonlinear algebraic equation $k_b^{(n)}(T) = k_T$.

The new stochastic piecewise differentiable on $t \ge t_0$ boundary trajectories $k_b^{(n)}$, $q_b^{(n)}$ under the new control $k_b^{(n)} = 1$ are solved by one of the numerical methods [6,12] from the stochastic system (3) in stochastic initial conditions with prehistories $k(y) = k_m(y)$ and $q(y) = q_m(y)$, $y \in [\zeta_2 - \tau, \zeta_2]$. Average new boundary trajectories $k_{OP}^{(n,c)}$ and $q_{OP}^{(n,c)}$ are calculated as $k_{OP}^{(n,c)}(t) = Mk_{OP}^{(n)}(t)$ and $q_{OP}^{(n,c)}(t) = Mq_{OP}^{(n)}(t)$, $t \ge t_0$.

5. According to the results of [8, 13], the stochastic and mean optimal process $\Pi_{OP} = \{k_{OP}(t), q_{OP}(t), u_{OP}(t-\tau),\ t \in [t_0, T]\}$ is the gluing at the moment of switching the control ζ_1 of the stochastic and mean boundary process Π_b with the stochastic and mean main process Π_m and gluing at the moment of switching the control ζ_2 of this stochastic and mean main process Π_m with a new stochastic and mean boundary process $\Pi_b^{(n)}$, i.e.

$$k_{OP}(t) = \begin{cases} k_b(t) \text{ at } t \in [t_0, \zeta_1], \\ k_m(t) \text{ at } t \in [\zeta_1, \zeta_2], \\ k_b^{(n)}(t) \text{ at } t \in [\zeta_2, T], \end{cases} \quad q_{OP}(t) = \begin{cases} q_b(t) \text{ at } t \in [t_0, \zeta_1], \\ q_b(t) \text{ at } t \in [\zeta_1, \zeta_2], \\ q_b^{(n)}(t) \text{ at } t \in [\zeta_2, T], \end{cases} \quad u_{OP}(t) = \begin{cases} u_b(t) \text{ at } t \in [t_0, \zeta_1], \\ u_m(t) \text{ at } t \in [\zeta_1, \zeta_2], \\ u_b^{(n)}(t) \text{ at } t \in [\zeta_2, T], \end{cases} \quad t \in [t_0, T].$$

Moreover, the optimal control by the capital accumulation rate u_{OP} is a piecewise continuous function on $t \in [t_0, T]$, and the optimal trajectories by the specific capital k_{OP} and by the specific volume of investment in scientific and technological progress q_{OP} are are piecewise differentiable functions on $[t_0, T]$. There can be at least two and at most four optimal processes Π_{OP}. In addition, the optimal control u_{OP} and control switching moments ζ_1 and ζ_2 are deterministic values, the optimal trajectories are stochastic.

Output from the algorithm.

Algorithm for calculating the optimal process when choosing the main process at the initial stage

1. Select stochastic and average main mode (process) $\Pi_m = \{k_m(t), q_m(t), u_m(t-\tau),\ t \ge t_0\}$ when performing equality in the phase plane kOq $\Phi(Mk_0(y), Mq_0(y)) = 2Mk_0(y) - 2Mq_0(y) + (l_1 - l_2) = 0$, $y \in [t_0 - \tau, t_0]$.

The economic system moves along the main trajectories by the specific capital k_m and by the specific volume of investments in scientific and technological progress q_m under the chosen main control u_m (in the phase plane kOq along a special straight curve (main line) $\Phi(k,q)=0$) until the moment ζ f descent from the special straight line.

2. The moment ζ is the moment of control switching and the greatest root (solution) of the equation $\Phi(Mk_m(t), Mq_m(t)) = 2Mk_m(t) - 2Mq_m(t) + (l_1 - l_2) = 0$, $t \ge t_0$, is calculated by the search

method among the solutions $\Phi(k,q)=0$, $t \ge t_0$ and the search for the greatest. Note that equality $\Phi(Mk_0(y),Mq_0(y))=0$ can be satisfied on some interval $[t_0-\tau,y_1]$, that is part of the segment $[t_0-\tau,t_0]$ or at one point $y=t_0-\tau$.

3. At the moment ζ the economic system leaves the special straight line $\Phi(k,q)=0$ and moves along a new boundary regime (process) $\Pi_b^{(n)}=\{k_b^{(n)}(t),q_b^{(n)}(t),u_b^{(n)}(t-\tau)=1,\ t \ge \xi\}$ to the moment T (the final moment of the study of the economic process) and which is determined by one of the numerical methods [6, 10, 11] from the nonlinear algebraic equation $Mk_b^{(n)}(T)=k_T$. The new stochastic boundary trajectories $k_b^{(n)}$ and $q_b^{(n)}$ under the boundary control $u_b^{(n)}=1$ are identified by one of the numerical methods [6, 12] from the system of stochastic dynamics of the movement of specific capital and specific volume of investments in scientific and technological progress (3) under stochastic initial conditions with prehistories $k(y)=k_m(y)$ and $q(y)=q_m(y)$, $y\in[\zeta-\tau,\zeta]$. The average boundary trajectories are identified as $k_b^{(n,c)}(t)=Mk_b^{(n)}(t)$ and $q_b^{(n,c)}(t)=Mq_b^{(n)}(t)$.

Note that this stochastic initial task has a piecewise differentiable unique solution on $t\in[\zeta,T]$ $(k_b^{(n)}(t),q_b^{(n)}(t))$, $t \ge \zeta$ in the sense of stochastic equivalence [6, 5, 14], as the function s is is piecewise continuous on $t \ge t_0$, the functions $k(y)$ and $q(y)$ are piecewise differentiable on $[\zeta-\tau,\zeta]$, the function $A(Q\ge 0)>0$ is twice continuously differentiable, monotonically increasing and concave, the function $f(k\ge 0)\ge 0$ is twice continuously differentiable monotonically increasing and concave. The average boundary trajectories $k_b^{(n,c)}(t)$ and $q_b^{(n,c)}(t)$ are given as $k_b^{(n,c)}(t)=Mk_b^{(n)}(t)$ and $q_b^{(n,c)}(t)=Mq_b^{(n)}(t)$, $t\ge t_0$.

4. The stochastic and average optimal process $\Pi_{OP}=\{k_{OP}(t),q_{OP}(t),u_{OP}(t-\tau),\ t\ge t_0\}$ according to the results of [8] is the gluing at the moment of control switching ζ stochastic and average main mode (process) $\Pi_m=\{k_m(t),q_m(t),u_m(t-\tau),\ t\ge t_0\}$ and stochastic and average new boundary mode (process) $\Pi_b^{(n)}=\{k_b^{(n)}(t),q_b^{(n)}(t),u_b^{(n)}(t-\tau)=1,\ t\in[\zeta,T]\}$.

Moreover, the optimal control by the capital accumulation rate u_{OP} is a piecewise differentiable function on $[t_0,T]$, and the optimal trajectories by the specific capital k_{OP} and the specific volume of investment n scientific and technological progress q_{OP} are are piecewise differentiable functions on $[t_0,T]$. In addition, there can be at least two and no more than four optimal modes (processes) Π_{OP}. Optimal controls by the rate of capital accumulation u_{OP} and control switching moments ζ_1, ζ_2 are deterministic values, and optimal trajectories by the specific capital k_{OP} and by the specific volume of investments in scientific and technological progress q_{OP} are stochastic. Exit from the algorithm.

Thus, the behavior of optimal trajectories is typical: if $K_0<K_T$ (respectively $k_0<k_T$), then at first the trajectories enter the special straight line (main line) $\Phi(k,q)=2k-2q+(l_1-l_2)=0$ and and move along it, after meeting with the control switching straight line, the trajectories leave the special straight line and move to the given final state k_T (respectively K_T). The economic interpretation of the special straight line is as follows: only on the special straight line the rate of efficiency of accumulation is equal to the rate of efficiency of investment in science.

Thus, for optimal management in this model we have: if the goal is sufficiently distant (K_0 much less than K_T), then all investments should be directed to the area where the rate of efficiency is the highest. On the main line (special straight line), investments are distributed in such a proportion that the efficiency rates are equal.

The above is formulated in the form of a theorem.

Theorem. Let the economic indicators for the stochastic model (3)—(7) satisfy the conditions:

1) $\mu \in (0;1)$, $n > 0$, $a \in (0;1)$, $b \in (0;1)$, $v \in (0;2)$, $t_0 \geq 0$, $T > t_0$, $\alpha_1, \beta_1, \alpha_2, \beta_2, L_0 > 0$, $k_T > 0$ are constant;

2) the function $s \in [0,1]$ — is piecewise continuous on $[t_0, T]$;

3) random functions k_0 and q_0 are piecewise continuous on $[t_0 - \tau, t_0]$;

4) macro-production function $f(k \geq 0) \geq 0$ is twice continuously differentiable, monotonically increasing and concave;

5) the multiplier of scientific and technological progress $A(Q \geq 0) > 0$ is a twice continuously differentiable, monotonically increasing concave function;

6) the solution of the system of nonlinear equations (11)—(12) exists.

Then the stochastic economic-mathematical model (3)—(7) has an optimal process. Moreover, the optimal control by the capital accumulation rate is a piecewise continuous function on $[t_0, T]$, and the optimal trajectories by the specific capital and by the specific volume of investment in scientific and technological progress are piecewise differentiable functions on $[t_0, T]$. In addition, the stochastic economic-mathematical model (3)—(7) can have at least two and no more than four optimal processes.

In stochastic modelling, it is necessary to know the confidence intervals for a given probability of the mean values and variances of normal general sets of optimal trajectories by specific capital and by the specific volume of investment in scientific and technological progress.

Let us consider a computation experiment to determine the optimal trajectories by the specific capital k_{OP} and by the specific volume of investment in scientific and technological progress q_{OP}, and have obtained N ensembles of the specific capital $k_{OP}^{(i)}(t)$, $t \geq t_0$, $i = \overline{1,N}$ and the specific volume of investment in scientific and technological progress $q_{OP}^{(i)}(t)$, $t \geq t_0$, $i = \overline{1,N}$.

We have calculated the sample statistics:

- sample averages for specific capital and specific volume of investments in scientific and technological progress $\overline{k}_{OP}(t) = N^{-1} \sum_{i=1}^{N} k_{OP}^{(i)}(t)$, $\overline{q}_{OP}(t) = N^{-1} \sum_{i=1}^{N} q_{OP}^{(i)}(t)$, $t \geq t_0$;
- sample variances for specific capital and specific volume of investments in scientific and technological progress $S_{k_{OP}}^2(t) = (N-1)^{-1} \sum_{i=1}^{N} \left(k_{OP}^{(i)}(t) - \overline{k}_{OP}(t) \right)^2$, $S_{q_{OP}}^2(t) = (N-1)^{-1} \sum_{i=1}^{N} \left(q_{OP}^{(i)}(t) - \overline{q}_{OP}(t) \right)^2$, $t \geq t_0$.

It should be noted that the sample averages are equal (coincide) to the average values of the optimal trajectories in terms of specific capital and specific volume of investments in science, defined above, i.e. $\overline{k}_{OP}(t) = k_{OP}^{(c)}(t)$, б $\overline{q}_{OP}(t) = q_{OP}^{(c)}(t)$, $t \geq t_0$.

Confidence intervals for a given probability $\theta \in (0,1)$ for the specific capital and specific volume of investments in scientific and technological progress are $\left(\frac{(N-1) S_{k_{OP}}^2(t)}{\chi_{1-\theta/2}^2 (N-1)} ; \frac{(N-1) S_{k_{OP}}^2(t)}{\chi_{\theta/2}^2 (N-1)} \right)$,

$\left(\frac{(N-1)S_{q_{OP}}^2(t)}{\chi_{1-\theta/2}^2(N-1)}; \frac{(N-1)S_{q_{OP}}^2(t)}{\chi_{\theta/2}^2(N-1)} \right)$, $t \geq t_0$, where $\chi_{\theta/2}^2(N-1)\left[\chi_1^2 - \theta/2(N-1)\right] - \theta/2\left[1-\theta/2\right]$ — quantile of the Pearson distribution χ^2 with $(N-1)$ degrees of freedom at the confidence level. Then the confidence limits for a given probability $\theta \in (0,1)$ or real values of specific capital and specific volume of investments in science look like

$$k_{OP}^{(p)}(t) \in \left(k_{OP}^{(c)}(t) - \frac{t_\theta (N-1) S_{k_{OP}}(t)}{\sqrt{N}}; k_{OP}^{(c)}(t) + \frac{t_\theta (N-1) S_{k_{OP}}(t)}{\sqrt{N}} \right),$$

$$q_{OP}^{(p)}(t) \in \left(q_{OP}^{(c)}(t) - \frac{t_\theta (N-1) S_{q_{OP}}(t)}{\sqrt{N}}; q_{OP}^{(c)}(t) + \frac{t_\theta (N-1) S_{q_{OP}}(t)}{\sqrt{N}} \right),$$

where $t_\theta - \theta$ — is the quantile of the two-sided Student's distribution with $(N-1)$ degrees of freedom at a given confidence level $\theta \in (0,1)$.

Remarks. The above is the case for the stochastic economic-mathematical model (3)—(7) when piecewise continuous functions α_1, α_2, β_1 and β_2 on $t \geq t_0$.

Conclusions

1. A stochastic model of single-product macroeconomics of growth with endogenous scientific and technological progress under Wiener and Poisson processes is proposed and its study is carried out.

2. The proposed stochastic economic-mathematical model has at least two and no more than four optimal processes.

3. For the proposed stochastic model, the optimal control by the rate of capital accumulation and the moments of control switching are deterministic, and the optimal trajectories by the specific capital and the specific volume of investments in scientific and technological progress are stochastic. The confidence intervals for a given probability for the real values of optimal trajectories by specific capital and by specific volume of investments in science are given.

References

[1] Ponomarenko, O.I., Perestyuk, M.O., & Burim, V.M. (1995). *Osnovy matematychnoyi ekonomiky [Fundamentals of mathematical economics].* Kyiv: Informatyka [in Ukrainian].

[2] Boychuk, M.V. & Semchuk, A.R. (2015). Modelyuvannya odnoproduktovoyi makroekonomiky zrostannya z endohennym tekhnichnym prohresom z urakhuvannyam spozhyvannya [Modeling of single-product macroeconomics of growth with endogenous technical progress taking into account consumption]. *Visnyk Chernivets'koho torhovel'no-ekonomichnoho instytutu – Bulletin of the Chernivtsi Trade and Economic Institute Chernivtsi, III (59),* 176–188 [in Ukrainian].

[3] Boychuk, M.V. & Semchuk, A.R. (2015). Stohasticheskaya model optimalnoy odnoproduktovoy ekonomiki rosta pri nelineynom ekologo-ekonomicheskom kriterii s vinerovskimi i puassonovskimi protsessami [A Stochastic Model of an Optimal Single-Product Growth Economy with a Nonlinear Ecological-Economic Criterion with Wiener and Poisson Processess]. *Problemyi upravleniya i informatiki – Problems of management and informatics, 3,* 136–148 [in Ukrainian].

[4] Boychuk, M.V. & Semchuk, A.R. (2012). *Modelyuvannya ta optymizatsiya povnoho tsyklu odnoproduktovoyi makroekonomiky zrostannya z urakhuvannyam ekolohichnoho faktora [Modeling and optimization of the full cycle of single-product macroeconomics of growth taking into account the environmental factor].* Chernivtsi: «Misto» [in Ukrainian].

[5] Skorokhod, A.V. (1990). *Lektsiyi z teoriyi vypadkovykh protsesiv [Lectures on the theory of random processes].* Kyiv: Lybid [in Ukrainian].

[6] Yasinskiy, V.K. (2005). *Osnovy obchislyuvalnih metodiv [Fundamentals of computational methods].* Chernivtsi: «Zoloti lytavry» [in Ukrainian].

[7] Kolmogorov A.N. & Fomin, S.V. (1968). *Elementyi teorii funktsiy i funktsionalnogo analiza [Elements of the theory of functions and functional analysis].* Moskva: Nauka [in Russia].
[8] Andreeva, E.A., Kolmanovskiy, E.A. & Shayhet, L.E. (1992). *Upravlenie sistemami s posledeystviem [Systems management with aftereffect].* Moskva: Nauka [in Russian].
[9] Elsgolts, L.E. & Norkin, S.B. (1971). *Vvedenie v teoriyu differentsialnyih uravneniy s otklonyayuschimsya [Introduction to the theory of differential equations with deviating argument].* Moskva: Nauka [in Russian].
[10] Beyko, I.V., Bublik, I.V. & . Zinko, P.N. (1983). *Metodyi i algoritmyi resheniya zadach optimizatsii [Methods and algorithms for solving optimization problems].* Kyiv: Vischa shkola [in Ukrainian].
[11] Vasilev, F.P. (1980). *Chislennyie metodyi resheniya ekstremalnyih zadach [Numerical methods for solving extremal problems]*. Moskva: Nauka [in Russian].
[12] Nikitin, N.N. (1978). Metodyi tsifrovogo modelirovaniya stohasticheskih differentsialnyih uravneniy i otsenka ih pogreshnosti [Methods for digital modeling of stochastic differential equations and estimation of their error]. *Zhurnal vyichislitelnoy matematiki i matematicheskoy fiziki – Journal of Computational Mathematics and Mathematical Physics, 1,* 106–117 [in Russian].
[13] Hryhorkiv, V.S. (2011). *Optymalne keruvannya v ekonomitsi [Optimal management in the economy]*. Chernivtsi: ChernIvetskiy nats. un-t [in Ukrainian].
[14] Yasinskiy, V.K. (2003). *Stokhastychni dynamichno-funktsional'ni systemy zi vsiyeyu peredistoriyeyu [Stochastic dynamic functional systems with all the background].* Kyiv: TViMS [in Ukrainian].

СТОХАСТИЧНЕ МОДЕЛЮВАННЯ ОДНОПРОДУКТОВОЇ МАКРОЕКОНОМІКИ ЗРОСТАННЯ З ЕНДОГЕННИМ НАУКОВО-ТЕХНІЧНИМ ПРОГРЕСОМ ПРИ ІНВЕСТИЦІЙНОМУ ЗАПІЗНЕННІ

Бойчук М.В., Вінничук О.Ю., Скращук Л.В.

Анотація

Вплив науково-технічного прогресу на характер зростання в економічній системі виявляється в різних формах, зокрема при розгляді неавтономних, тобто змінних у часі макровиробничих функцій. Неврахування деяких економічних показників у економіко-математичних моделях, невизначеність та неточність вхідної інформації приводить до стохастичного моделювання при дослідженні економічних систем і процесів.

У даній статті розглядається стохастична економіко-математична модель із використанням вінерівських і пуассонівських процесів, де вплив наукових досліджень на виробництво запрограмований в самій системі, тобто в моделі з внутрішнім (ендогенним) врахуванням науково-технічного прогресу. Тому актуальним, як у теоретичному, так і практичному плані є вплив наукових досліджень на виробництво в самій ендогенній системі та ще з інвестиційним запізненням.

Запропонована стохастична модель однопродуктової економіки зростання з ендогенним науково-технічним прогресом при інвестиційному запізненні з вінерівськими та пуассонівськими процесами. В стохастичній економіко-математичній моделі враховано, що кінцевий випуск продукції використовується на споживання, на капіталовкладення в розширення основних фондів, на покращення виробництва з урахуванням ефективності затрат «на науку», на оподаткування, на урядові витрати, на сальдо та на ліквідацію забруднення навколишнього середовища. Ця модель враховує вплив наукових досліджень на економічне виробництво у самій системі, тобто цей вплив є ендогенною змінною. Для дослідження стохастичної оптимальної еколого-економічної моделі використано стохастичні достатні умови оптимальності.

При дослідженні стохастичної економіко-математичної моделі побудовано алгоритм розрахунку оптимального процесу при виборі необхідного економічного крайового режиму на

початковій стадії, а також алгоритм розрахунку оптимального процесу при виборі економічного магістрального режиму при виборі на початковій стадії.

Література

1. Пономаренко О.І. Перестюк О.І., Бурим В.М. Основи математичної економіки. Київ : Інформатика, 1995. 320 с.
2. Бойчук М.В., Семчук А.Р. Моделювання однопродуктової макроекономіки зростання з ендогенним технічним прогресом з урахуванням споживання. *Вісник Чернівецького торговельно-економічного інституту. Економічні науки*. Чернівці, 2015. Вип. ІІІ (59). С. 176–188.
3. Бойчук М.В., Семчук А.Р. Стохастическая модель оптимальной однопродуктовой экономики роста при нелинейном эколого-экономическом критерии с винеровскими и пуассоновскими процессами. *Проблемы управления и информатики.* Київ, 2015. №3. С. 136–148.
4. Бойчук М.В., Семчук А.Р. Моделювання та оптимізація повного циклу однопродуктової макроекономіки зростання з урахуванням екологічного фактора. Чернівці : Місто. 2012. 208 с.
5. Скороход А.В. Лекції з теорії випадкових процесів. Київ : Либідь, 1990. 168 с.
6. Ясинський В.К. Основи обчислювальних методів. Чернівці : Золоті литаври, 2005. 396 с.
7. Колмогоров А.Н., Фомин С.В. Элементы теории функций и функционального анализа. Москва : Наука, 1968. 496 с.
8. Андреева Е.А., Колмановский В.Б., Шайхет Л.Е. Управление системами с последействием. Москва : Наука, 1992. 336 с.
9. Эльсгольц Л.Э., Норкин С.Б. Введение в теорию дифференциальных уравнений с отклоняющимся аргументом. Москва : Наука, 1971. 296 с.
10. Бейко И.В., Бублтк Б.Н.. Зиньк П.Н. Методы и алгоритмы решения задач оптимизации. К. : Вища школа, 1983. 512 с.
11. Васильев Ф.П. Численные методы решения экстремальных задач. Москва : Наука, 1980. 518 с.
12. Никитин Н.Н. Методы цифрового моделирования стохастических дифференциальных уравнений и оценка их погрешности. *Журнал вычислительной математики и математической физики*, 1978. Т. 18, №1. С. 106–117.
13. Григорків В.С. Оптимальне керування в економіці. Чернівці : Чернівецький нац. ун-т, 2011. 200 с.
14. Ясинський В.К. Стохастичні динамічно-функціональні системи зі всією передісторією. Київ : ТВіМС, 2003. 304 с.