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RESEARCH OF THE BALLISTIC FLIGHT TRAJECTORY OF A SOLID BODY OF SPHERICAL AND CYLINDRICAL FORM TAKING AIR RESISTANCE INTO ACCOUNT

ДОСЛІДЖЕННЯ БАЛІСТИЧНОЇ ТРАЄКТОРІЇ ПОЛЬОТУ ТВЕРДОГОТІЛА СФЕРИЧНОЇ ТА ЦИЛІНДРИЧНОЇ ФОРМИ З УРАХУВАННЯМ ОПОРУ ПОВІТРЯ

The article presents the results of modeling and research of ballistic flight trajectories of spherical and cylindrical solid bodies taking into account air resistance based on linear and quadratic dependence. Ballistic flight trajectories were calculated for different values of the initial speed and throwing angles, and a safety paraboloid was constructed. Based on the results of the study, it was established that the main parameters affecting the flight range are the value of the initial flight speed of the solid body and the value of the throwing angle. It is also determined that for each case of a flight range shorter than the maximum achievable, there are two flight trajectories with different values of the throw angle.
***Keywords**: mathematical modeling of flight, ballistic trajectory, Stokes drag, Newton drag.*

Актуальність даної статті обумовлена необхідністю удосконалення математичних моделей польоту твердих тіл сферичної і циліндричної форми таподальшого розвитку методів дослідження їх балістичних траєкторій.

Метою статті є побудова моделі польоту твердого тіла сферичної та циліндричної форми у просторі з урахуванням різних варіантів моделювання сили лобового опору повітря.

Проведено дослідження балістичних траєкторій польоту твердого тіла з моделюванням лобового опору повітря лінійною залежністю при малих швидкостях (опір Стокса) та квадратичною — при великих швидкостях (опір Ньютона). Математична модель задачі побудована на основі другого закону Ньютона та представлена у вигляді диференціальних рівнянь руху тіла в проекціях наосікоординат. Побудова розв’язку для випадку квадратичної залежності сили лобового опору повітря має особливість врахування напрямку польоту тіла. Для фази підйому тіла на максимальну висоту врахування сили лобового опору повітря зменшує дальність польоту та збільшує її на фазі спуску.

Проведено дослідження траєкторії польоту твердого тіла циліндричної з квадратичним опором повітря з різними кутами кидання та початковими швидкостями та побудовано параболоїд безпеки. За результатами дослідження було встановлено, що основними параметрами впливу на дальність польоту являються величина початкової швидкості польоту твердого тіла та значення кута кидання. Також визначено, що для кожного випадку дальності польоту меншої максимально досяжної існує дві траєкторії польоту з різними значеннями кута кидання.

В подальших дослідження необхідно побудувати моделі та провести дослідження балістичних траєкторій із урахуванням впливу атмосферних та погодних умов.

***Ключові слова**: математичне моделювання польоту, балістична траєкторія, опір Стокса, опір Ньютона.*

Problem's Formulation

One of the main properties of aerodynamics, which must be taken into account when modeling the flight of solid bodies, is the consideration of air resistance. Construction and research of mathematical models of such problems is important from both a scientific and an applied point of view. This makes it possible to adequately assess the impact of aerodynamic forces on the object's flight, which is important when designing aircraft and artillery ammunition. Therefore, further development of research methods of mathematical models of ballistic flight trajectories of solids of different shapes is quite relevant.

Analysis of recent research and publications

A few scientific works are dedicated to the investigation of various aspects of mathematical models for studying the ballistic flight trajectories of solid bodies. For instance, in the paper [1], the results of three possible trajectory approximations are provided: a low-angle trajectory, a trajectory with a large launch angle, and a trajectory where the horizontal velocity is approximately equal to the vertical velocity. Closed-form solutions for the range of the first option are obtained using Lambert's W-function. It was also noted that the range of trajectories for the considered cases is symmetric when the launch angle is close to $\pi/4$, while in other cases, the trajectories are asymmetric. In [2], explicit analytical expressions for calculating ballistic trajectories are constructed, allowing for the accurate determination of the maximum height and the time of its attainment, as well as the flight range of the body at the highest ascent. An important feature of this approach is its lack of restrictions on initial velocity, launch angle, and the drag coefficient of the medium. Publication [3] discusses the application of the Laplace decomposition method (LDM) for obtaining an approximate calculation of the two-dimensional motion of a rigid body with linear air resistance. Using this method, an approximate solution to the ballistic problem is derived, considering nonlinear dependencies. The issues of constructing a mathematical model and increasing the accuracy of ballistic trajectory calculations for long flight ranges are addressed in [4]. The construction of a mathematical model and the investigation of the Magnus force's influence due to the nutation angle on the projectile trajectory are presented in the article [5]. However, the comprehensive impact of the initial parameters considering the air resistance force has not been fully explored.

Formulation of the study purpose

Building a mathematical model and conducting a study of the ballistic flight trajectory of a spherical and cylindrical solid body for various initial parameters taking into account air resistance.

Presenting main material

The problem of modeling and calculating the flight of a solid body with velocity v and angle φ between the velocity vector and the horizon line is considered. The origin of the coordinate system is placed at the starting point of the body's flight, with the y-axis directed vertically upwards and the x-axis directed horizontally, as shown in Fig. 1.

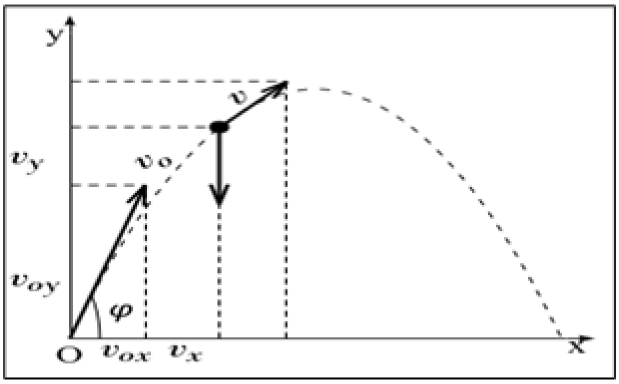

Fig. 1. Scheme of calculating the ballistic flight trajectory

During the motion of a solid body, the velocity is directed tangent to the trajectory of motion, and its projections onto the coordinate axes are determined by the following relationships:

$$v_x = v\,cos\varphi;\quad v_y = v\,sin\varphi. \tag{1}$$

The initial velocity at $t = 0$ will be equal to v_0, and the angle φ — is equal to φ_0. The force F acting on the body during the flight will also be expressed as projections onto the Ox, Oy axes:

$$F_x = Fcos\varphi;\quad F_y = Fsin\varphi. \tag{2}$$

The mathematical model of the motion of a solid body along a ballistic trajectory in projections onto the coordinate axes will be expressed as follows:

$$\left.\begin{aligned} m\frac{dv_x}{dt} &= F_x \\ m\frac{dv_y}{dt} &= F_y \end{aligned}\right\}. \tag{3}$$

Let's note that during the flight of a solid body along a ballistic trajectory, it is subject to the force of gravity and the force of air resistance, i.e., $F = F_g + F_{air}$. The curvature and rotation of the Earth are not taken into account. The force of gravity is constant and always directed downwards. The force of air resistance is always directed opposite to the direction of motion in the surrounding medium and its magnitude depends on the absolute velocity and the shape of the body:

$$F_{air} = -k \cdot f(v), \tag{4}$$

where k — empirical coefficient depending on the parameters of the body and the medium.

At low speeds, where the Reynolds number $Re< 1$, the dependence of the air resistance force on the velocity of the solid body $f(v)$ is linear (Stokes' drag). For spherical bodies, it is defined by the following relationship [6]:

$$F_{air} = -6\pi\nu d \cdot v, \tag{5}$$

where ν — the kinematic viscosity of the medium, d — the diameter of the body.

Thus, the equation of motion of a solid body of spherical shape in the direction of the coordinate axes, taking into account the force of gravity and the Stokes drag force, is as follows:

$$\left.\begin{aligned} m\frac{dv_x}{dt} &= -kv_x \\ m\frac{dv_y}{dt} &= -mg - kv_y \end{aligned}\right\}, \tag{6}$$

where $k = 6\pi\nu d$.

For convenience, let's divide both sides of the equation by m:

$$\left.\begin{aligned} \frac{dv_x}{dt} &= -\frac{k}{m}v_x \\ \frac{dv_y}{dt} &= -g - \frac{k}{m}v_y \end{aligned}\right\}. \tag{7}$$

The initial conditions for the equations (6) of the flight of a spherical body are determined by the initial parameters at the starting moment:

$$t = 0:\ v_x(0) = v_0\,cos\,\varphi, \tag{8}$$

$$v_y(0) = v_0\,sin\,\varphi. \tag{9}$$

Solving the system of equations (7) taking into account the initial conditions (8), (9), we obtain:

$$\left.\begin{aligned} v_x(t) &= v_0 cos\varphi e^{-\frac{kt}{m}} \\ v_y(t) &= -\frac{gm}{k} + (v_0 sin\varphi + \frac{gm}{k})e^{-\frac{kt}{m}} \end{aligned}\right\}. \tag{10}$$

Displacement along the coordinate axes x, y is determined by the system of equations:

$$\frac{dx}{dt} = v_x;\ \frac{dy}{dt} = v_y. \tag{11}$$

Let's substitute the expressions for velocities v_x, v_x from (10) into equation (11), and we'll obtain the differential equations to determine the trajectory of the flight of the spherical body:

$$\left.\begin{aligned} \frac{dx}{dt} &= e^{-\frac{kt}{m}} v_0 cos\varphi \\ \frac{dy}{dt} &= -\frac{gm}{k} + (v_0 sin\varphi + \frac{gm}{k})e^{-\frac{kt}{m}} \end{aligned}\right\}. \tag{12}$$

For equations (12), the initial conditions are determined by the body's position at the origin of the coordinate system at the starting time:

$$\text{At}\, t = 0:\ x = 0,\ y = 0. \tag{13}$$

By integrating equations (12) with consideration of the initial conditions (13), we obtain the equation of the trajectory of the flight of the spherical body in parametric form:

$$\left.\begin{array}{c} x=\frac{mv_0 cos\varphi}{k}\left(1-e^{-\frac{kt}{m}}\right) \\ y=\frac{m}{k}\left(\frac{mg}{k}+v_0 sin\varphi-gt-\frac{gme^{-\frac{kt}{m}}}{k}-e^{-\frac{kt}{m}}v_0 sin\varphi\right) \end{array}\right\}. \tag{14}$$

At Reynolds numbers $Re >> 1$, the dependency of air resistance on velocity is quadratic (Newtonian drag) and is defined as follows [6]:

$$F_{air} = -sign(v) \cdot \frac{1}{2} c_d A\rho \cdot v^2, \tag{15}$$

where c_d — the coefficient of aerodynamic drag ($c_d = 0.47$ for spherical bodies, $c_d = 0.82$ for cylindrical bodies), A — the cross-sectional area of the body, ρ — is the density of the medium, $sign(v)$ — the sign of the velocity of the body.

During the flight of a solid body, the force of frontal air resistance is directed in the opposite direction of the body's movement, and accordingly, in the mathematical model, during the ascent phase, it has a negative sign along the y-axis, and during the descent phase, it has a positive sign. Therefore, to account for this peculiarity in the trajectory of the flight, during the ascent phase, we will use equations with air resistance directed downwards (y_1), and during the descent phase, upwards (y_2). Thus, the system of equations describing the motion of the solid body with consideration of Newtonian air resistance will be expressed as follows:

$$\left.\begin{array}{c} m\frac{dv_x}{dt}=-kv_x{}^2 \\ m\frac{dv_{y1}}{dt}=-mg-kv_{y1}{}^2 \\ m\frac{dv_{y2}}{dt}=-mg+kv_{y2}{}^2 \end{array}\right\}, \tag{16}$$

where $k = \frac{1}{2} c_d A\rho$.

The solution to the system of equations (16) considering the initial conditions (8), (9) will be expressed as:

$$\left.\begin{array}{c} v_x(t)=\frac{mv_0 cos\varphi}{ktv_0 cos\varphi + m} \\ v_{y1}(t)=\sqrt{\frac{mg}{k}}tg\left(arctg\left(\frac{v_0 sin\varphi\sqrt{k}}{\sqrt{gm}}\right)-\sqrt{\frac{kg}{m}}t\right) \\ v_{y2}(t) = \sqrt{\frac{mg}{k}}\frac{\left(\left|\frac{\sqrt{\frac{mg}{k}}+v_0 sin\varphi}{\sqrt{\frac{mg}{k}}-v_0 sin\varphi}\right|\cdot e^{-2t\sqrt{\frac{gk}{m}}}-1\right)}{\left|\frac{\sqrt{\frac{mg}{k}}+v_0 sin\varphi}{\sqrt{\frac{mg}{k}}-v_0 sin\varphi}\right|\cdot e^{-2t\sqrt{\frac{gk}{m}}}+1} \end{array}\right\}. \tag{17}$$

Accordingly, the ballistic trajectory of the solid body's flight will be determined by the following system of equations:

$$\left.\begin{array}{c} \frac{dx}{dt}=\frac{mv_0 cos\varphi}{ktv_0 cos\varphi + m} \\ \frac{dy_1}{dt}=\sqrt{\frac{mg}{k}}tg\left(arctg\left(\frac{v_0 sin\varphi\sqrt{\frac{k}{m}}}{\sqrt{g}}\right)-\sqrt{\frac{kg}{m}}t\right) \\ \frac{dy_2}{dt} = \sqrt{\frac{mg}{k}}\frac{\left(\left|\frac{\sqrt{\frac{mg}{k}}+v_0 sin\varphi}{\sqrt{\frac{mg}{k}}-v_0 sin\varphi}\right|\cdot e^{-2t\sqrt{\frac{gk}{m}}}-1\right)}{\left|\frac{\sqrt{\frac{mg}{k}}+v_0 sin\varphi}{\sqrt{\frac{mg}{k}}-v_0 sin\varphi}\right|\cdot e^{-2t\sqrt{\frac{gk}{m}}}+1} \end{array}\right\}. \tag{18}$$

By integrating equations (18) with consideration of the initial conditions (13), we obtain the equation of the trajectory of the flight of the solid body in parametric form:

$$\left.\begin{array}{c} x=\frac{m}{k}ln\left(\frac{ktv_0cos\varphi}{m}+1\right) \\ y_1=\frac{m}{k}ln\left(\left|\frac{cos\left(arctg\left(\frac{v_0sin\varphi\sqrt{\frac{k}{m}}}{\sqrt{g}}\right)-t\sqrt{\frac{kg}{m}}\right)}{cos\left(arctg\left(\frac{v_0sin\varphi\sqrt{k}}{\sqrt{gm}}\right)\right)}\right|\right) \\ y_2=\sqrt{\frac{mg}{k}}t+ \\ +\frac{m}{k}ln\left(\frac{\left|\frac{\sqrt{\frac{mg}{k}}+v_0sin\varphi}{\sqrt{\frac{mg}{k}}-v_0sin\varphi}\right|\cdot e^{-2t\sqrt{\frac{gk}{m}}}+e^{-2t\sqrt{\frac{gk}{m}}}}{\left|\frac{\sqrt{\frac{mg}{k}}+v_0sin\varphi}{\sqrt{\frac{mg}{k}}-v_0sin\varphi}\right|\cdot e^{-2t\sqrt{\frac{gk}{m}}}+1}\right) \end{array}\right\}. \tag{19}$$

In this case, the continuity of the flight trajectory during the ascent and descent phases is ensured by the conditions: when $t = t_{h,\max}$: $y_1 = y_2$; $\frac{dy_1}{dt} = \frac{dy_2}{dt} = 0$.

The time it takes for the body to ascend along the trajectory to its maximum height is determined as follows:

$$t_{h,max} = \frac{\sqrt{m}\cdot \mathrm{arctg}\left(\frac{v_0 \sin\varphi\sqrt{\frac{k}{m}}}{\sqrt{g}}\right)}{\sqrt{kg}}. \tag{20}$$

Using the functions described above, we will construct the flight trajectory of solid bodies of spherical and cylindrical shapes. Assuming that the flight of the cylindrical body is gyroscopically stable, based on the expressions (19), we obtain the dependencies of flight trajectories for various initial parameters shown in Fig. 2, 4. The results of calculating the flight trajectories of a spherical body using expressions (14) for different values of initial velocity and launch angles are presented in Fig. 3.

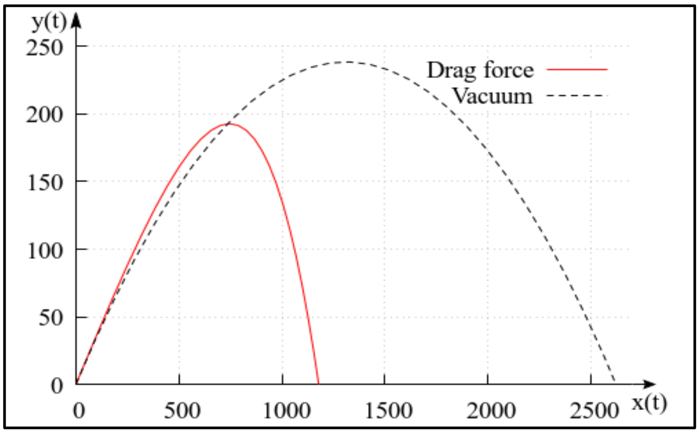

Fig. 2. Flight trajectories of a cylindrical body with and without consideration of air resistance force at an initial velocity of 200 m/s

The analysis of the results presented in Fig. 3 and 4 shows that as the initial velocity of the cylindrical body increases, the angle required to achieve the maximum flight range decreases. At high velocities, this value becomes slightly less than π/4, which ensures maximum flight range without considering air resistance. With increasing velocity, the influence of horizontal velocity on the flight range increases.

The results of calculating the ballistic trajectories of the cylindrical solid body for various initial velocities and different launch angles allow us to construct a paraboloid of ballistic trajectories known as the safety paraboloid, presented in Fig. 5.

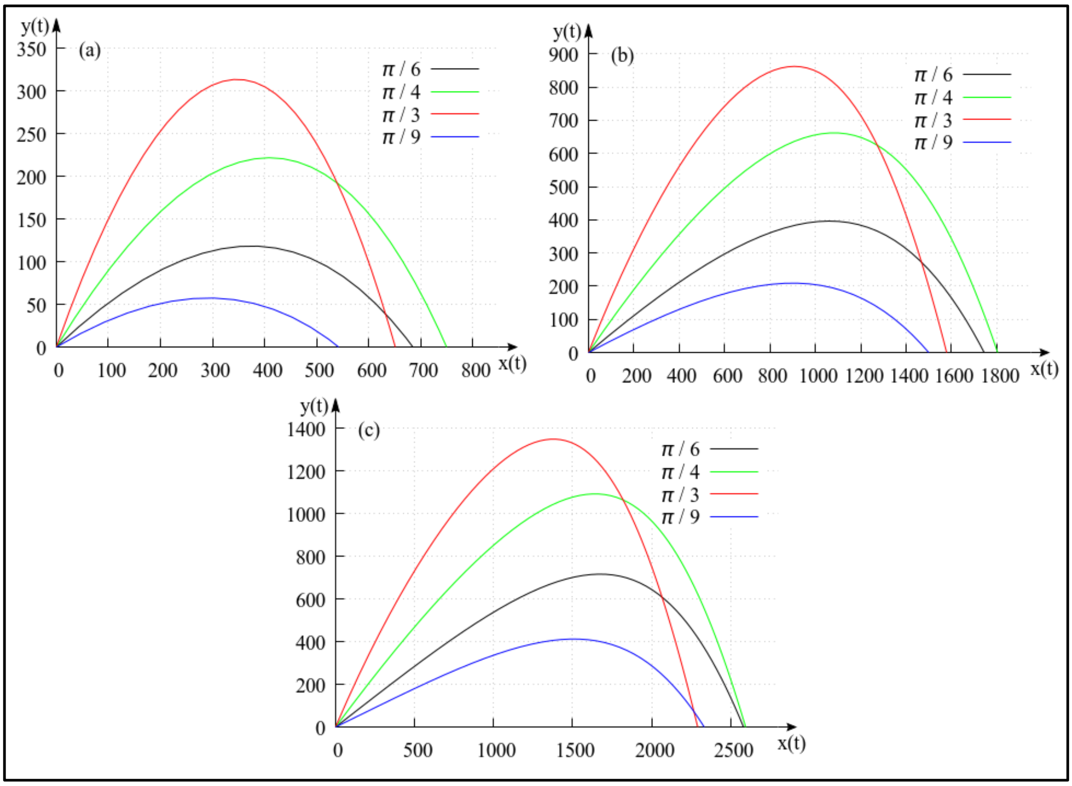

Fig. 3. Flight trajectories of a spherical body with different launch angles φ and initial velocities (a) — 100 m/s, (b) — 200 m/s, and (c) — 300 m/s

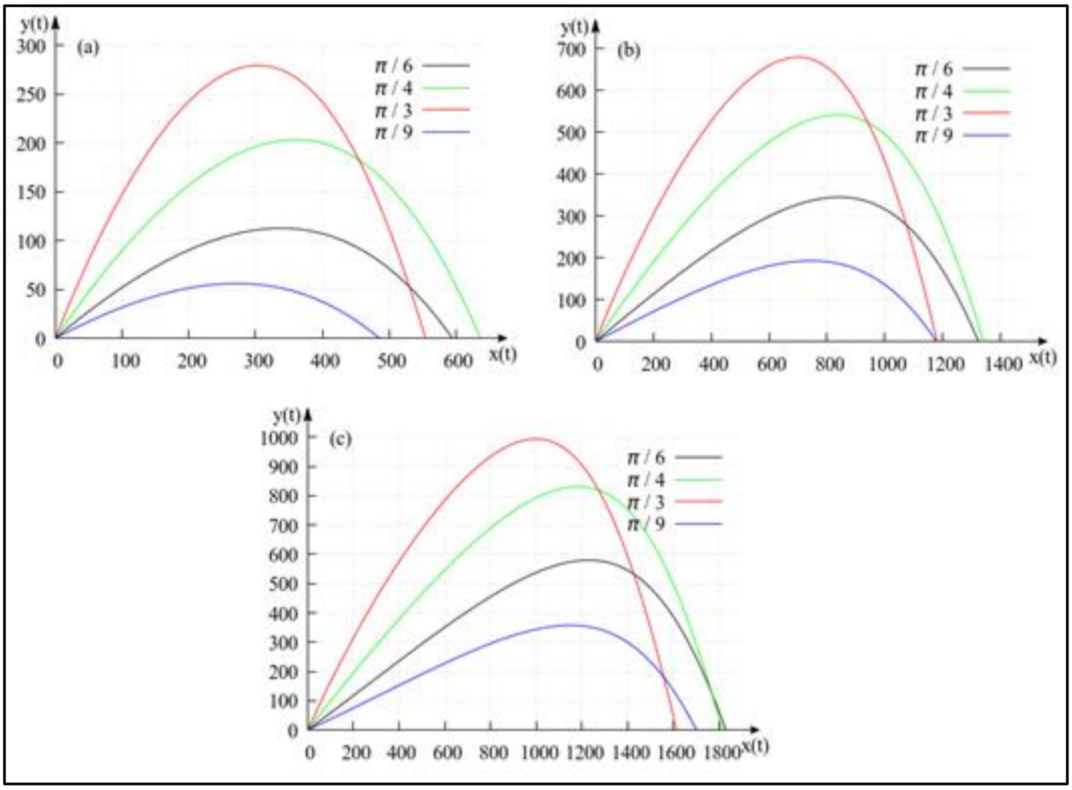

Fig. 4. Flight trajectories of a cylindrical body with different launch angles φ and initial velocities (a) — 100 m/s, (b) — 200 m/s, and (c) — 300 m/s

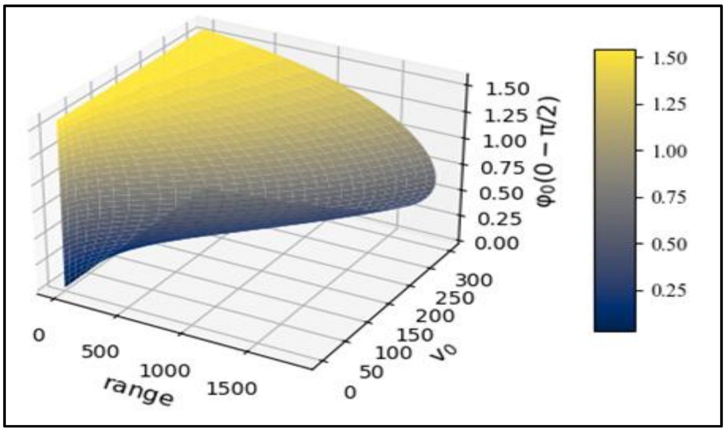

Fig. 5. Paraboloid of ballistic trajectories of a solid cylindrical body flight

Conclusions

The analysis of the conducted research shows that the obtained analytical solutions provide a comprehensive understanding of the trajectory of motion of a solid body and allow for the investigation of the dependence of flight trajectories for various initial conditions of the problem. The flight range of a solid body is primarily determined by the parameters of the initial velocity and the launch angle. The solution for the case of quadratic dependence of the air resistance force has the peculiarity of taking into account the direction of the body's flight. For the ascent phase of the body to its maximum height, considering the air resistance force reduces the flight range, while it increases during the descent phase. It is also noted that for each case of flight range smaller than the maximum achievable, there are two flight trajectories with different launch angles.

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