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FEATURES OF MATHEMATICAL MODELING OF SOME INVERSE PROBLEMS IN GEOPHYSICS

ОСОБЛИВОСТІ МАТЕМАТИЧНОГО МОДЕЛЮВАННЯ ДЕЯКИХ ЗВОРОТНИХ ЗАДАЧ ГЕОФІЗИКИ

The article conducts a mathematical study of some aspects of the phenomenon of mobile voids that can cause landslides. This class includes natural phenomena — karst craters, the cause of which is the geological structure and groundwater. In addition, there are similar phenomena that are man-made. They occur in mining enterprises — mines and quarries. In particular, great difficulties arise in the process of open-pit mining of iron ore in quarries. Sometimes local voids appear in the walls of quarries for geological reasons. Under the influence of explosions, they move to the surface and can cause catastrophic landslides. These phenomena are united by the fact that they are difficult to predict. And after a landslide, it is not clear whether further landslides are possible?

At present, there are no effective means for solving such problems. Of all the methods of geophysics, the most suitable for solving such problems are gravimetric methods. The aim of the research is to create methods and algorithms for detecting potentially dangerous moving cavities based on observation of the gravitational field by systematic measurements using high-precision gravimeters. The authors proposed methods for building a system for detecting the presence of moving cavities and determining their parameters. Based on these algorithms, numerical experiments were conducted that significantly reduce the risk of such disasters. In addition, in the event of local cavities, their parameters can be established, which will allow diagnostic drilling — the ultimate means of diagnosing the presence of dangerous cavities.

***Keywords**: filtration; smoothing; least squares method; ill-posed problem; integral equation of the first kind; ill-conditioned problems; computational experiment.*

У статті проводиться математичне дослідження деяких аспектів явища рухомих пустот, які можуть викликати обвали. До цього класу зокрема належать природні явища — карстові воронки, причиною яких є геологічна структура та підземні води.

Крім того, існують аналогічні явища, що мають техногенний характер. Вони мають місце на гірничих підприємствах — шахтах та кар'єрах. Тут головну роль відіграють технологічні фактори, перш за все вибухи для добування корисних копалин. Обставиною, що істотно обтяжує становище для шахтних підприємств є порушення технологій видобутку руди в минулому, коли вибирались найбільш дешеві способи рекультивації відпрацьованих забоїв.

Аналогічні труднощі виникають в процесі видобутку залізної руди відкритим способом в кар'єрах. Цей процес супроводжується постійними, ще більш потужними технологічними вибухами Іноді в стінках кар'єрів з геологічних причин виникають локальні пустоти. Під впливом вибухів вони рухаються до поверхні і можуть спричиняти катастрофічні обвали. Всі ці три явища об'єднує той факт, що їх важко передбачити. А після обвалу не зрозуміло чи можливі подальші обвали?

На сьогоднішній момент не існує ефективних засобів для вирішення подібних задач. З усіх методів геофізики найбільш підхожими для вирішення подібних проблем є методи гравіметрії. Метою дослідження є створення методів та алгоритмів для виявлення потенційно небезпечних рухомих порожнин на основі спостереження за гравітаційним полем шляхом систематичних вимірів за допомогою використання високоточних гравіметрів.

Авторами запропонований методи побудови системи для виявлення факту наявності рухомих порожнин та визначення їх параметрів. На базі цих алгоритмів проведенні чисельні експерименти, що дозволяють істотно знизити ризик подібних катастроф. Крім того, в разі виникнення локальних порожнин можна встановити їх параметри, що дозволить виконати діагностичне буріння — остаточний засіб діагностики наявності небезпечних порожнин.

***Ключові слова**: фільтрація; згладжування; метод найменших квадратів; некоректна задача; інтегральні рівняння першого роду; погано обумовлені задачі; обчислювальний експеримент.*

Problem's Formulation

Systematic landslides have become a commonplace phenomenon in many countries around the world [1]. Often, these are sinkholes that occur naturally. They are caused by the processes of dissolution and mechanical erosion of soil by groundwater. Such phenomena can threaten buildings, hydraulic structures, and cause difficulties in agriculture. At the initial stage of their formation, they can lead to deformation of structures. This usually leads to their destruction. There are also cases of destruction of transport infrastructure, railways and highways, bridges and tunnels. In addition, such phenomena are dangerous for mining companies. There have been cases of mine flooding and destruction of mine structures. This phenomenon leads to large losses and even human casualties.

The most dangerous thing is that there is no reliable way to predict the occurrence of such geological phenomena. Moreover, it is impossible to assert that the evolution of a sinkhole is complete and it is in a stationary phase, i.e., it is not expanding.

Similar phenomena, but mainly of anthropogenic nature, occur in mining enterprises, mines and quarries. Here, technological factors play a major role, primarily explosions for mining. In particular, the city of Kryvyi Rih, which has a long mining history, is notable for such disasters. The case of violations of iron ore mining technologies in the past.

he local press in Kryvyi Rih reports specific cases of such accidents. The most recent example is a sinkhole over a hundred meters deep that formed on 29.03.24. The process has not stabilized to this day. It is difficult to estimate the depth of the sinkhole because its edges are constantly collapsing. A similar landslide occurred in 2010 in the yard of a shopping center.

In the same year, a sinkhole was created in a mine as a result of blasting. One person died when he fell into the pit with his car.

Similar difficulties arise in the process of open pit mining. The fact is that this process is also accompanied by constant technological explosions. And they are much more powerful than in mines. As a result, local voids sometimes appear in the sides (walls) of quarries for geological reasons. Under the influence of explosions, they move to the surface and cause catastrophic collapses. This causes large economic losses and can lead to human casualties. All three phenomena are united by the fact that they occur suddenly at the stage of collapse, not expected. And after the collapse, it is not clear whether the destructive process is over or not. In other words, are further collapses possible? This raises several questions.

First, to prevent a possible collapse. This means being able to detect potentially dangerous voids at the earliest possible stage of development, underground, when they are still moving towards the surface. This way, it is possible to limit the presence of people and equipment in the area of possible collapse and thus minimize economic losses. For this purpose, it is highly desirable to be able to determine the approximate geometry and volume of the dangerous cavity.

Secondly, if a collapse has already occurred, one must be able to answer the question: has it already ended, or is the process continuing deep down and further development of the failure possible? At the moment, there are no effective tools for solving such problems.

Of all the methods of geology, the most suitable for solving these problems are the methods of gravity exploration, the oldest and most versatile part of geophysics.

The purpose of the research is to create methods for detecting potentially dangerous moving cavities based on systematic observation of the gravitational field by using high-precision gravimeters. The authors propose methods for building a system to detect the presence of moving cavities and determine their parameters.

Analysis of recent research and publications

Theoretically, the problem of localizing any cavities in the domain P can be solved using the three-dimensional gravimetry equation [2]:

$$\int_P (z - z_J) \cdot \rho(x, y, z)/R^3 \cdot dV = F_Z(M_S)/G\,, \qquad (1)$$

where $\rho(M)$ — density of the medium at point M with coordinates x, y, z, where $M \in P$, R — distance between point M, and M_S, G — gravitational steel. M_S in (1) denotes the observation point whose coordinates are known. It usually lies on the Earth's surface. It measures the vertical component of gravity F_Z. Choosing the number and location of observation points is at the discretion of the system user. So, based on this information, it is necessary to identify areas of voids, that is, it is necessary to determine the areas where the volume density $\rho(M)$ close to zero.

Problems of the type of equation (1) are known in mathematics. This is an integral equation of the first kind [2—3]. These problems belong to the so-called inverse problems and are ill-conditioned [3—5]. Typical and very important problems of this class are the study of the structure of an object by the secondary fields generated by the object. For example, the inverse problem of magnetostatics, where it is necessary to find the distribution of the magnetization vector in this object given the external magnetic field of a ferromagnetic object. This problem is very important, but it is characterized by the fact that sometimes it does not have an unambiguous solution.

The solution of (1) in general is very difficult. These equations are very difficult to solve because they are poorly conditioned. The fact is that the inverse nonlinear integral operator is degenerate, or almost degenerate, and at least very sensitive to errors in the measurement of the gravitational field [4]. This leads to large calculation errors even with small errors in gravitational field measurements. To solve such problems correctly, it is necessary to filter the input data and smooth them.

Therefore, equation (1) is always solved in practice with various simplifications [6,7]. In practice, such a way to solve the problem is very difficult at the present stage of development of geophysics.

Formulation of the study purpose

Objective of the research is to mathematically model the process of localization of moving voids (MV). This includes, first of all, a confident statement of the presence of MV.

In addition, if an MV exists, questions about the parameters of the MV, such as its location and volume, must be answered. Based on the results of the study of MV parameters, diagnostic drilling is performed. This procedure is very expensive; one such well costs about 50 thousand dollars.

It is difficult and often dangerous to perform, as there is a possibility of a collapse.

The authors propose a much simpler and more reliable algorithm to answer important questions about MV research. Let us replace the solution of equation (1) with the study of a plane unsteady vector field, where each point of the plane is associated with a vector — value of the normal gravity (NG). It depends both on the coordinates of the measurement point and on time. Measurements at certain moments of time NG provides a discrete analog of this field. Based on this information, it is necessary to answer the main question that has already been formulated: is there an MV in the area under study? This question will be studied on the example of a mining quarry, because in cases of sinkholes and collapses of old mines, funding is minimal for various reasons, resources are unsatisfactory, and it is difficult to apply the proposed methods. In these cases, systematic NG measurements are usually not performed. And in the case of quarries, there is a full-time service that researches this issue.

Presenting main material

Obviously, if the vector field NG is stationary, then no MV exist and there is no danger of collapse. But almost always this field is non-stationary. This is not always due to the presence of significant MV.

Additional reasons for the non-stationarity of NG, in addition to the existence of MV, are:

1. Gravimeter measurement errors. This is the most significant cause of NG non-stationarity.

2. Fluctuations in the regional background. This phenomenon is caused by the movement of large masses of rock at considerable depths, about several kilometers or more, inside the Earth. The regional background fluctuation will be the same at all measurement points and it is relatively small, but not zero, and must be taken into account.

3. Much smaller errors than the influence of the regional background on the results of gravity measurements can be caused by heavy precipitation (rain and snow), after which the density of the topsoil can change.

4. Theoretically, ore extraction in an open pit has a very minor impact on the change in NG. This factor is very small.

The numerical values of factors 3 and 4 are much smaller than the error of the gravimeter, so we will not take them into account.

Let's write down all the factors that affect the gravimeter readings at a given point with the number *S*, at time *T*:

$$FN_S^T = C_S + P_S^T + PG_S^T + R^T . \quad (2)$$

Here through FN_S^T value of NG at the moment of time *T*, at the point with the number *S*, C_S — value NG at this point depends on the constant elements of the geological structure, it depends only on the observation point, and not on time, P_S^T — value NG at this point from moving cavities, PG_S^T — absolute error of the gravimeter at time *T* at the point with the number *S*. In addition, NG is influenced by R^T — the regional background, we believe that it does not depend on the coordinates of the point and changes only over time

Only the maximum absolute error of the gravimeter PG_S^T is known, which is very small for modern gravimeters. We will assume that this error is distributed according to a normal law [6].

It should be noted that gravimetry is constantly improving its parameters. For example, serial gravimeters from the Canadian company SCINTREX are capable of providing an accuracy of up to 1 microgal, and modern differential GPS provide an accuracy of up to 1 centimeter. This allows you to automatically operate very accurately with the coordinates of the measurement points.

But as mathematical modeling shows, even gravimeters of the previous generation with a measurement accuracy one or two orders of magnitude lower than the new models can quite acceptably solve the problem of localizing medium-sized moving cavities.

A certain problem is the assessment of the regional gravity background. Its fluctuations can significantly exceed the error of the gravimeter. This can cause an erroneous response of the system, it will show the presence of MV where they do not exist.

Let the following be selected: the number of observation points and their coordinates, the given accuracy of the gravimeter, and the known average rock density in the quarry. Also assume that there is a special, carefully selected so-called base point, which lies in an area where a collapse is not possible. It should be chosen as far away from the edge of the quarry as possible. We will assume that the impact of potential MVs at this point is minimal.

It is needed to estimate the regional background, which should be assessed as accurately as possible.

Namely:

$$FN_B^T = C_B + P_B^T + PG_B^T + R^T , \; FN_B^{T+1} = C_B + P_B^{T+1} + PG_B^{T+1} + R^{T+1} ;$$

$\Delta FN_B^T = FN_B^{T+1} - FN_B^T = P_B^{T+1} - P_B^T + PG_B^{T+1} - PG_B^T + R^{T+1} - R^T$. то $\Delta FN_B^T \approx \Delta R_B^T + \Delta PG_B^T$, since $P_B^{T+1} \approx P_B^T$, because, given the choice of the reference point, potential moving cavities have virtually no effect on the gravimeter readings. To some extent, this estimate can be distorted by the gravimeter error. As you know, in the worst case, the errors of the first difference, ΔPG_B^T, can double. Therefore, it is advisable to take measurements at the base point $K > 1$ times and find the average value. Then, according to the theory of errors, the average value will be $\sqrt{K}$ times more accurate than a single measurement. In this case, the value of ΔPG_B^T can be neglected, and then we have a fairly accurate estimate for the regional background fluctuation:

$$\Delta FN_B^T \approx \Delta R_B^T. \qquad (3)$$

Similarly to the previous one, at any point S we can construct a finite difference

$$\Delta FN_S^T = FN_S^{T+1} - FN_S^T = P_S^{T+1} - P_S^T + PG_S^{T+1} - PG_S^T + R^{T+1} - R^T.$$

Then

$$\Delta FN_S^T = \Delta P_S^T + \Delta PG_S^T + \Delta R^T. \qquad (4)$$

Obviously, if there is no MV at the moment, then $\Delta FN_S^T = \Delta PG_S^T + \Delta R^T$ і $\Delta FN_S^T \approx \Delta R^T$. We believe that the regional background fluctuations are reliably estimated in formula (3). Based on (4), we can formulate the criteria for the presence of MV.

If the value of (4) exceeds several times (more than 2—3) the error of the gravimeter, then there are grounds for the presence of MV near this point. Such points will be called suspicious for the existence of MV. If there are several such points and they are located close to each other, there are much more reasons for alarm. In this case, it is necessary to increase the number of measurements near the possible MV and maximize the accuracy of measurements — make them several times and average the results, if possible.

When the existence of the MV is fixed, it can be considered a ball at considerable depths with great accuracy [6], then it is necessary to determine its parameters: the coordinates of the center and its radius. We will assume that at the moment there is only one MV in the field of study. In practice, this is always the case.

To refine the MV parameters, solve the following problem. There are N fixed observation points on the Earth's surface of the region P, whose coordinates are known. Let us denote:

$$Q_S = 3 \cdot F_Z(x_S, y_S, z_S)/(4 \cdot \pi \cdot \rho \cdot G),$$

here ρ — average ore density in the open pit, G — gravitational constant, $F_Z(x_S, y_S, z_S)$ — NG at the point S. Next, let us sign x_K, y_K, z_K coordinates of the center of the equivalent MV ball, R_K its radius, through R_S — distance between the center of the ball and the observation point S.

Suppose there are M suspicious points, and $M > 4$. Now we will select only suspicious points as observation points. Thus, we need to solve the system of nonlinear equations:

$$(z_K - z_S) \cdot R_K^3 / R_S^3 = Q_S, \qquad (6)$$

assuming the coordinates of the center of the sphere and its radius are unknown, $S = 1, M$. The number of equations K in (6) can be large. Therefore, (6) will be solved by the method of least squares [7].

It should be noted that the phenomena analyzed in this research do not occur often. They are very difficult to predict, and in some cases, it is dangerous to investigate directly because of a possible collapse. There are examples of people dying as a result of an attempt to measure in a dangerous place. Therefore, mathematical modeling is crucial in the research of the dynamics of moving cavities. The main parameters on which the quality of mathematical modeling depends are the error of the gravimeter and the location of the observation points, i.e. the points where gravity is measured. In addition, you need to have information about the average density of the rock in the quarry. The latter information is always available from geological observations. MV parameters are also an important factor in mathematical modeling. The success of the proposed algorithms and software depends on the quality and completeness of mathematical modeling.

Thus, the main tasks of mathematical modeling were as follows:

1) determination of the presence of MV in researching area;

2) determination of MV parameters.

The absolute error of the gravimeter in the mathematical modeling could take the values of 30, 20 and 10 microgals. They were chosen for the reason that the gravimeter with a measurement accuracy of 30 microgals was used at the mining enterprise where the research was conducted,

Other values of this parameter were chosen for the reasons of purchasing a more efficient device and analyzing the impact of improving this parameter on the efficiency of the system.

It is important to describe in a few words the simplified structure of the quarry in terms of rational selection of observation points for mathematical modeling. It is a very large depression in the Earth, in our case 330 meters. Its walls are very steep. It is dangerous to approach the edge of the quarry closer than a few meters. The surface of the pit can be considered flat, at least near the side. It is also important to note that MVs occur in the quarry wall (side). The choice of observation points near the wall is very difficult, but this is where the most dangerous places in terms of collapse and the most accurate observations are.

In addition, the ore is mined in a relatively compact area and process explosions occur in the vicinity of the mine. Therefore, it is most appropriate to choose observation points near this place. However, it should be noted that the distance from the measurement point to the center of the potential MV at the beginning of its evolution may be large. Only large cavities can be detected in this way, but they are the ones that pose the greatest danger. A case of an MV with a diameter of more than 35 meters was recorded at the enterprise.

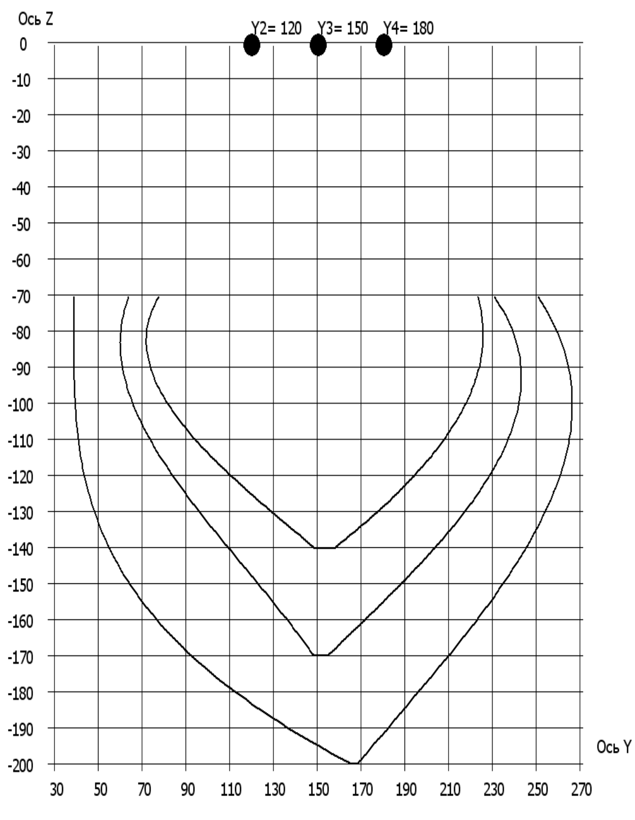

Fig 1. Areas of reliable determination of the parameters of the MV

The main result of the numerical modeling of the MV search is shown in Fig. 1.

It shows the areas of almost exact determination of the parameters of the spherical cavity at three different values of the absolute error of the gravimeter parameters [8,9].

In this case, the observation system consists of points located on three parallel lines. These are the points where the gravimeter readings are abnormal. Let's call them active points. Only they were considered in the system (6), other points were ignored. In total, there were 21 active observation points. In all numerical experiments, a local cavity was studied, which was a sphere with a radius of 22 meters. The areas of reliable detection are parabolic in shape. The area with the smallest area (upper) corresponds to the gravimeter with an accuracy of 30 microgals, the middle one 20 and the lowest one 10 microgals. The latter has the largest area, which is natural. The zero level was formed using five measurements made with a short time interval and averaged to reduce the measurement error.

Thus, for a gravimeter with an error of 10 microgal, the coordinates of the center of gravity with a radius of 22 m can be reliably found by the system 170 meters from the surface and below. This circumstance is crucial for conducting so-called diagnostic boreholes. As mentioned earlier, each such well costs about 50 thousand dollars. To confidently diagnose MV, several such wells need to be performed. Thus, the final decision on the presence of MV can cost about 200 thousand dollars. Incorrect operation of the system can lead to the collapse of the quarry side, destruction of equipment, and loss of life. In this case, losses are very difficult to predict.

Conclusions

Authors have developed a method for detecting moving cavities based on systematic measurements of the gravitational field in a special system of points. In addition, an efficient algorithm for calculating the parameters of moving cavities is proposed. Numerical experiments have been conducted to study the effectiveness of the algorithms for detecting moving cavities for several values of gravimeter error. On the basis of the proposed algorithms, a method for calculating the geometry of the region, at each point of which it would be possible to reliably determine all the parameters of the moving cavity, is implemented.

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