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OPTIMIZATION OF THE ALGORITHM FOR CALCULATION OF THE MAGNETOSTATICS PROBLEM FOR AREAS OF A SPECIAL TYPE

ОПТИМІЗАЦІЯ АЛГОРИТМУ РОЗРАХУНКУ ЗАДАЧІ МАГНІТОСТАТИКИ ДЛЯ ОБЛАСТЕЙ СПЕЦІАЛЬНОГО ВИДУ

Effective calculation of electromagnetic devices is a crucial condition for mathematical modeling of complex physical systems. At the moment, this task is far from over. In particular, the problem of modeling magnetostatic systems (MS) cannot be considered solved. Let us recall that MCs are physical devices, the primary sources of the magnetic field for which are either permanent magnets, or stationary currents, or a combination of such sources. Currently, MCs make up a significant part of modern devices. Therefore, the task of detailed modeling of the operation of such systems is very important. The aim of the study is "approximately" axisymmetric systems, and they have a three-dimensional zone, the size of which is very small relative to the entire MS. But it is in this zone that the magnetic field needs to be found, and the calculation of the field needs to be done very precisely. Therefore, the calculation of such systems has specific features and is associated with significant difficulties. Such MCs can be called quasi-horizontal (KVS). These systems are of great practical importance, in particular, they are widely used in electronic optics, and have a number of other applications.

In order to perform the precessional calculation of the KVS, the authors proposed an algorithm that logically consists of two stages. At the first stage, the MS is calculated as fully axisymmetric. At the second stage, only the three-dimensional working part of the MS is calculated, the results of the first stage are used as an external field.

***Keywords**: magnetic systems; method of vector integral equations; modeling of electronic-optical devices; convergence of the iterative process; non-linear environments.*

Автори досліджують особливості розрахунку для важливого класу магнітостатичних систем, що поєднують в собі властивості як тривимірних так і вісесиметричних. Практично весь прилад вісесиметричний, за виключенням робочої тривимірної області. Ці системи дуже важкі для чисельного аналізу, бо, як правило, вони вимагають практично прецизійного розрахунку саме в тривимірній зоні.

Зокрема, для приладів електронної оптики магнітне поле формує конфігурацію потоку електронів, невірний розрахунок якого призводить до значного підвищення температури пристрою і навіть до можливої зміни магнітних властивостей матеріалу приладу. Крім того, ефективність роботи приладу в цьому випадку значно знижується.

Основна проблема розрахунку таких магнітостатичних систем пов'язана з дискретизацією. Для подібних систем вона в значній мірі вимушено не рівномірна. Це викликає проблеми із збіжністю нелінійного ітераційного процесу. А саме, якщо виконати дискретизацію магнітостатичної системи відносно рівномірно, то матимемо недопустимо велику кількість елементів, час розрахунку таких систем буде дуже великим. А в випадку не рівномірної дискретизації, як показують чисельні експерименти, ітераційний процес може бути розбіжним. Також слід звернути увагу на те, що вплив вісесиметричної частини приладу на тривимірну падає залежно від відстані до умовної осі приладу. Як правило, тривимірна частина розташована близько від осі обертання і має незначний об'єм. Причому, для цієї частини потрібно визначити розподілення якомога точніше. Вісесиметрична частина може розглядатись в цих умовах як джерело первинного магнітного поля. Як показують чисельні експерименти,похибки в розподіленні вектора намагніченості від цієї частини практично не дуже істотно впливають на основний процес — на розрахунок поля в тривимірній частині.

Таким чином, розрахунок подібних систем має істотні особливості і пов'язаний зі значними специфічними труднощами. Такі МС можна назвати квазівісесмиметричними (КВС). Їх можна виділити в окремий підклас тривимірних магнітних систем. Подібні прилади використовуються в електронній оптиці, та мають інші важливі застосування.

***Ключові слова**: магнітні системи; метод векторних інтегральних рівнянь; моделювання електронно-оптичних приладів; збіжність ітераційного процесу; нелінійні середовища.*

Problem's Formulation

Efficient calculation of electromagnetic devices is a crucial condition for mathematical modeling of complex physical systems. At the moment, this task is far from being completed. In particular, the problem of modeling magnetostatic systems (MS) cannot be considered completely solved. Recall that MS are physical devices whose primary sources of magnetic field are either permanent magnets or stationary currents, or a combination of such sources. Currently, MSs make up a significant part of modern devices [1,2]. Therefore, the task of detailed modeling of such systems is very important. In particular, this includes the modeling of arbitrary nonlinear environments as a prerequisite for real devices [3].

In turn, the optimization of the designs of such systems implies the possibility of detailed modeling of their operation. One of the circumstances that significantly complicates modeling is that real MSs have a complex three-dimensional geometry that needs to be taken into account.

The location of the observation points (OPs) relative to the MS structure and the requirements for the accuracy of the magnetic field calculation are very important. Observation points are those points at which the magnetic field or other MS parameters are to be calculated. Specifically, if the OPs are located far from the MS surface, the calculation is relatively simple, but if not, a detailed approximation of the MS magnetization vector is required, at least in the vicinity of the OPs. This, in turn, raises problems with MS sampling and the convergence of the nonlinear iterative process. Namely, if the MS discretization is performed relatively uniformly, we will have an unacceptably large number of elements, and the calculation time of such systems will be very long. And in the case of non-uniform discretization, the iterative process may be divergent [4].

The purpose of this research is to consider "almost" axisymmetric magnetostatic systems that have a three-dimensional working zone, its size is very small relative to the entire MS. However, it is in this zone that the magnetic field must be found, and the field calculation must be done very accurately. Therefore, the calculation of such systems has important features and is associated with significant difficulties. We will call such MSs quasi-viscosymmetric (KVS) [5]. These systems are of great practical importance, in particular, they are widely used in electronic optics [6,7], and have a number of other important applications.

To perform the precession calculation of KVS, the authors propose an algorithm that logically consists of two stages. At the first stage, the MS is calculated as fully axisymmetric. At the second stage, only the three-dimensional working part of the MS is calculated, and the results of the first stage are used as an external field. This algorithm and its difficulties are described in detail in [5]. However, it should be noted that the three-dimensional and axisymmetric stages of the KVS calculation have a number of specific features that make them more efficient. This is crucial for the accuracy of the magnetic field calculation and program execution time.

Analysis of recent research and publications

As is well known, MS modeling methods can be divided into two main classes: differential and integral. Differential methods directly approximate the nonlinear differential equations of magnetostatics and boundary conditions arising from the problem statement. The most common method of this class is the finite element method (FEM) [8—10].

For integral methods, it is not necessary to set boundary conditions, and this is an important advantage of integral methods [4,5,11]. Integral methods for magnetostatics problems are formulated in the form of a nonlinear multidimensional equation with respect to physical or calculated field characteristics. The authors propose a method of vector integral equations (VIE) for physical field vectors as the main method for calculating the KVS [4,5,11]. It is advisable to use it for modeling most MSs, where this method has a potential advantage over the FEM. However, it should be noted that VIE is not sufficiently studied, both practically and theoretically, compared to the FEM.

Formulation of the study purpose

Purpose of this research is to improve the efficiency of calculating both the three-dimensional and axisymmetric parts of the KVS of magnetic systems [4,5]. Undoubtedly, this is the key to optimizing the KVS operation. The structure of both the three-dimensional and axisymmetric parts of such systems have important features that allow us to create effective specialized methods for calculating such MS. The objective of the study is to try to use the features of this structure to optimize the most resource-intensive calculations, i.e., the formation of a matrix of coefficients of a nonlinear system of equations. To do this, it is necessary to calculate a large number of integrals over a three-dimensional volume of a specific type, which takes almost all the calculation time. Note that the use of quadrature formulas for their calculation is less efficient than the proposed approach. In the three-dimensional case, the authors present analytical formulas for their calculation. This allows us to efficiently calculate the components of the tensor for a three-dimensional field from a homogeneously magnetized elementary polyhedron. If the polyhedron has a special shape, the formulas are greatly simplified. The algorithm for such a calculation is parallel, which makes it possible to further increase the calculation efficiency.

In addition, to calculate the KVS of the axisymmetric part, it is necessary to calculate the components of the tensor from a homogeneously magnetized toroidal element very accurately. Such an element usually has a rectangular cross-section. In this case, the components of the tensor are complete and incomplete elliptic integrals of the first, second, and third kind. They are not taken analytically. For this case, the authors propose effective approximation algorithms for their calculation.

Presenting main material

As described in [4,5], for the vector $\overline{U}(x)$, where $\overline{U}(x) = \overline{B}/\mu_0 + \overline{H} + 2(\overline{H}_0 + \overline{H}_M) + \overline{M}$, valid nonlinear equation (1):

$$\overline{U}(x) = 2 \cdot \overline{H}_0(x) + \int_V K(x,y) \cdot \overline{M}(\overline{U}(y) \cdot dv_y \,. \qquad (1)$$

From equation (1), we can obtain the magnetization vector M in the region V. To do this, let us divide the region of magnets V into elementary regions V_i, so that their union constitutes the region V. Moreover, no two regions intersect on a set of nonzero volume. We assume that in each elementary region V_i the magnetization is constant [4,5].

First, let's consider the MS calculation at the three-dimensional stage, since it requires the most time. For the calculation of the magnetic field, the efficiency of solving the discretized equation (1) is crucial. The practice of MS calculations has shown that it is most efficient to use hexagons as V_j for the three-dimensional case. The authors were able to reduce the calculation of the volume integral of the polyhedron V_j to the surface integral over its surface S_i. Thus, the key to solving equation (1) is to calculate the integral:

$$\overline{H} = 1/(4\pi)\int_{\Gamma} \overline{r}/r^3 ds = 1/(4\pi)\oint_{\Gamma} \mathrm{grad}(1/r) ds \,. \qquad (2)$$

It is advisable to optimize the calculation of (2) as much as possible. The integral (2) is calculated analytically in elementary functions if the boundary of the domain is rectilinear. In particular, it is convenient to perform calculations for an arbitrary quadrilateral face in the local coordinate system, where the *Y*-axis coincides with one of the diagonals and the *X*-axis is perpendicular to it. However, in the case of a rectangular face, it is more expedient to introduce another coordinate system with axes parallel to the sides of the rectangle, and the origin of the coordinate system coincides with one of the vertices of the rectangle. In this case, the integral (2) is calculated very efficiently. Let the rectangle have side lengths a and b, and the observation point $M_0(x_0,y_0,z_0)$ does not lie on the continuation of any side of the rectangle. Let's number the vertices of the rectangle: the point with coordinates (0,0) is 1, the point with coordinates (a,0) is 2, the point with coordinates (a,b) is 3, the point with coordinates (0,b) is 4. Let us denote by r_i the distance from the observation point M_0 to the point in the rectangle with number i. Then:

$$H_x = \ln[(b-y_0+r_3)\cdot(b-y_0+r_1)/((-y_0+r_2)\cdot(-y_0+r_4))];$$
$$H_y = \ln[(a-x_0+r_3)\cdot(-x_0+r_1)/((a-x_0+r_2)\cdot(-x_0+r_4))]. \tag{3}$$

If $z_0 = 0$ so $H_z = 0$, else:

$$H_z = z_0\cdot(H_{z1}+H_{z2}-H_{z3}-H_{z4}). \tag{4}$$

Here

$$H_{z1} = arxtg((b-y_0)\cdot(a-x_0)/(z_0\cdot r_3));$$
$$H_{z2} = arxtg(y_0\cdot(a-x_0)/(z_0\cdot r_2));$$
$$H_{z3} = arxtg((b-y_0)\cdot x_0)/(z_0\cdot r_4));$$
$$H_{z1} = arxtg(y_0\cdot x_0/(z_0\cdot r_1)).$$

It should be noted that the sum of arctangles in (4) can be reduced to the arctangent of the sum, only the formula in this case will be more complicated, but the calculations will be more efficient. For an irregular quadrilateral, formulas similar to (3) and (4) can also be given, but they are less commonly used and more complex. In addition, similar formulas can be given in both cases if the observation point lies on the extension of one of the sides of the quadrilateral.

Structure of formulas (3—4) and the features of KVS allow us to organize their calculation very efficiently. The optimization possibilities lie at the level of discretization, and now we will be interested only in the working three-dimensional area. The description of the MS geometry by the user usually consists in specifying the primary polyhedra, which are then automatically discretized into elementary ones. It should be noted that the number of primary polyhedra is relatively small, in the order of 10—30. Fig. 1 shows a typical discretization of half of the working area. Discretization of the original polyhedra into elementary ones consists in drawing the planes dividing the original polyhedron.

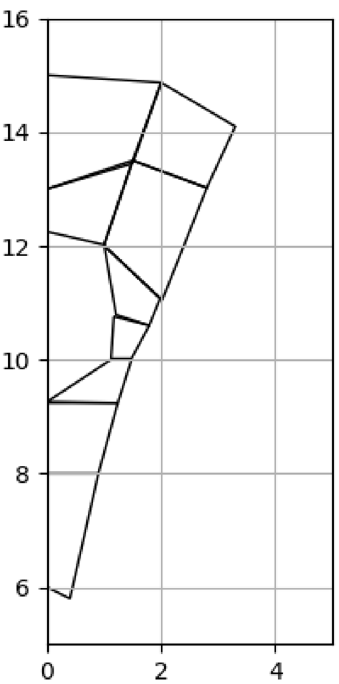

Fig. 1. Discretization of half of the working of the workspace into primary polyhedra

If we consider the problem of solving the discretized equation (1) not at the level of elementary polyhedra, but at the level of the original ones, then there are many opportunities to optimize the calculation of integrals (3) and (4). Namely, adjacent elementary polyhedra from the same original polyhedron have the same common faces. Therefore, integrals (3) and (4) for a common observation point can be calculated only once. This will almost double the time gain, since only the outer faces of the polyhedron do not give a gain. In addition, for a fixed initial polyhedron, it is possible to calculate the distances ri from a given observation point M_0 to the vertices of the elementary polyhedra and store them in an array. This will significantly reduce the time for calculating integrals (3) and (4), since the operation of calculating the root and logarithm takes a long time.

At the axisymmetric stage of the KVS calculation, it is necessary to be able to solve the problem of calculating the field for the case of an axisymmetric domain V as accurately as possible. To do this, let's calculate the elements of the tensor from equation (1) Let's consider the most important case of a toroidal element of rectangular cross-section. In this case, let us assume $\overline{M} = \bar{i}_Z M_Z + \bar{i}_R M_R$. Here $\bar{i}_Z$ — orth of the vertical axis, and $\bar{i}_R$ — orth of the horizontal axis, M_Z = const, $M_R = M/R$, where M = const. Then in this case $div\,\overline{M} = 0$ inside each element and the volume integrals in the discretized equation (1) can be reduced to surface integrals.

Let us denote the observation point by $Q(R_Q,Z_Q)$, is the point of integration of the volume V through $M(R_M,Z_M)$, $y = Z_Q - Z_M$. In addition, we denote the modulus of the integrals E and K by k: $k = 2\sqrt{R_Q R_M} / \sqrt{(R_Q + R_M)^2 + y^2}$, and the characterization of the integral of the third kind:

$$k_Q = 2\sqrt{R_Q R_M} /(R_Q + R_M).$$

Then the components of the tensor from an axisymmetric elementary toroid of rectangular cross section are equal to [11]:

$$H_{RR} = -M_R \cdot yk/(2R_Q)\sqrt{R_M / R_Q} \cdot [K(k) + (R_Q - R_M)/(R_Q + R_M) \cdot \Pi(-k_Q^2, k)]; \quad (5)$$

$$H_{RZ} = -M_Z \sqrt{R_M / R_Q} \cdot [1/k(2-k^2)K(k) - 2E(k)]; \quad (6)$$

$$H_{ZR} = -M_R \sqrt{R_M / R_Q} \cdot k \cdot K(k) \cdot E(k); \quad (7)$$

$$H_{ZZ} = -M_Z \cdot yk/\sqrt{R_M R_Q} \cdot [K(k) - (R_Q - R_M)/(R_Q + R_M) \cdot \Pi(-k_Q^2, k)]. \quad (8)$$

All components of the tensor must be multiplied by a constant $1/(2\pi\mu_0)$.

Formulas (5—8) are the original ones. That is, let the point (R_1,Z_1) is the lower left vertex of the rectangle, and the point (R_2,Z_2) respectively, the upper right. If the tensor component is calculated by the formula $F(R,Z)$, then the final value of the tensor will be equal to: $F(R_2,Z_2) - F(R_2,Z_1) - F(R_1,Z_2) + F(R_1,Z_1)$.

The observation point has fixed coordinates — (R_Q, Z_Q). The vertices of the rectangle M are successively the vertices of the rectangle: (R_1, Z_1), (R_2, Z_1). (R_2, Z_2), (R_1, Z_2).

In formulas (5—8), K, E, and Π are complete elliptic integrals of the first, second, and third kind. Recall that [12—14]:

$$K(k) = \int_0^{\pi/2} dt / \sqrt{1 - k^2 \sin^2 t}\,,\ E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t}\,;$$

$$\Pi(n,k) = \int_0^{\pi/2} dt/((1 - n\sin^2 t) \cdot \sqrt{1 - k^2 \sin^2 t}).$$

Biggest difficulty in formulas (5—8) is the calculation of the function $\Pi(-k_0^2, k)$.

In our case $\Pi(-k_0^2, k)$:

$$\Pi(-k_0^2, k) = \int_0^{\pi/2} dt/((1 + k_0^2 \sin^2 t) \cdot \sqrt{1 - k^2 \sin^2 t}).$$

Since $k_0^2 < k$, then this integral corresponds to the circular case, according to the classification [14]. After the transformations

$$H_{RR} = -M_R / 2 \cdot R_Q / R_M \cdot [y / P \cdot K(k) \cdot (R_Q - R_M)/(R_Q + R_M) + \pi / 2 + T)]; \quad (9)$$

$$H_{ZZ} = -M_Z \cdot R_Q / R_M \cdot [y / P \cdot K(k) \cdot (R_Q - R_M)/(R_Q + R_M) + \pi / 2 + T)]. \quad (10)$$

Here

$$P = \sqrt{(R_Q - R_M)^2 + (Z_Q - Z_Q)^2} ;$$

$$T = sign[y \cdot (R_Q - R_Z)] \cdot [K(k) \cdot F(\varphi, k_1) - K(k) \cdot E(\varphi, k_1) - K(k) \cdot E(k) \cdot F(\varphi, k_1)]. \quad (11)$$

To transform expressions (5) and (8), we used the lambda function of Heyman. Thus, we were able to reduce the calculation of the elliptic integral of the third kind to a linear combination of incomplete elliptic integrals of the first and second kinds, which are easier to calculate. Also, in formulas (9—11), we use complete elliptic integrals of the first and second kind. Now the main difficulties are caused by the calculation of $F(\varphi, k'), E(\varphi, k')$. These are incomplete elliptic integrals of the first and second kind:

$$F(\varphi, k') = \int_0^{\varphi} 1 / \sqrt{1 - (k')^2 \sin^2 t} dt, \ E(\varphi, k') = \int_0^{\varphi} \sqrt{1 - (k')^2 \sin^2 t} dt, \text{ де } k' = \sqrt{1 - k^2} .$$

The analysis of formulas (5—11) shows that the calculation of the field at the first stage depends on the complete and incomplete elliptic integrals calculated at the vertices of the rectangle. As in the three-dimensional case, it is very efficient to calculate these values only once, store them in a small auxiliary array, and only then directly find the coefficients of the nonlinear system of equations.

It is convenient to calculate full and incomplete elliptic integrals on the basis of the arithmetic-geometric mean Gauss [14].

Specifically, $A_I = (A_{I-1} + B_{I-1}) / 2, B_I = \sqrt{A_{I-1} \cdot B_{I-1}}, C_I = (A_{I-1} - B_{I-1}) / 2$. For the calculation of the complete elliptic integrals, we should put $A_0 = 1, B_0 = \sqrt{1 - k^2}, C_0 = 1$.As a result, we get: $K(k) = \pi / (2 \cdot A_N), E(k) = K(k) - 0.5 \cdot \sum_{i=1}^{N} 2^i \cdot C_i^2$. The value of N is selected from the condition $C_N < \delta$, where δ — accuracy of calculations.

To calculate the incomplete elliptic integrals, we should put $A_0 = 1, B_0 = k, C_0 = \sqrt{1 - k^2}$. In this case, the Landen transform is used [14]:

$$tg(\varphi_{N+1} - \varphi_N) = B_N / A_N \cdot tg(\varphi_N), \varphi_0 = \varphi.$$

As a result, we get:

$$F(\varphi, k') = \varphi_N / (2 \cdot A_N), \ E(\varphi, k') = \sum_{i=1}^{N} C_i \cdot \sin(\varphi_i) + E' / K' \cdot F(\varphi, k').$$

N is also selected from the condition $C_N < \delta$, where δ — accuracy of calculations.

Note that the use of quadrature formulas to calculate (5—8) is less efficient than the proposed approach. The case when R_M and R_Q go to infinity is especially unfavorable. Then the expressions for calculating H_{RR} and H_{ZZ} are of a type of uncertainty ∞ - ∞. In this case, k tends to one. This dramatically reduces the accuracy of the calculation using quadrature formulas.

Conclusions

1. Efficient algorithm for calculating the KVS magnetostatic fields using a nonlinear multidimensional integral equation has been implemented. This method consists of two stages — axisymmetric and three-dimensional.

2. Analytical calculation formulas for the three-dimensional KVS stage are derived.

3. Optimized calculations at the three-dimensional stage, which is the most time-consuming. This made it possible to reduce the calculation time of the three-dimensional stage for real problems by

almost 2 times. The speedup factor significantly depends on the KVS geometry and its discretization

4. Efficient computational algorithms for the components of the tensor from an axisymmetric elementary toroid of rectangular cross-section at the axisymmetric stage are proposed. This made it possible to significantly improve the calculation accuracy.

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