МОДЕЛЮВАННЯ ТА ОПТИМІЗАЦІЯ В ТЕХНОЛОГІЇ КОНСТРУКЦІЙНИХ МАТЕРІАЛІВ

SIMULATION AND OPTIMIZATION IN TECHNOLOGY OF CONSTRUCTION MATERIALS

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MATHEMATICAL MODEL FOR DESCRIPTION THE CREEP OF AGING CONCRETE USING CANONICAL POLYNOMIALS AND LANCZOS τ-METHOD

МАТЕМАТИЧНА МОДЕЛЬ ОПИСУ ПОВЗУЧОСТІ СТАРІЮЧОГО БЕТОНУ ЗА ДОПОМОГОЮ КАНОНІЧНИХ ПОЛІНОМІВ ТА τ-МЕТОДУ ЛАНЦОША

The article proposes a solution to the problem of rheological behavior of aging concrete, which takes into account both the effects of hereditary nature and the change in the elastic modulus over time, as well as the influence of the age of the material before loading on the ultimate creep deformations using canonical polynomials and the Lanczos method. The polynomials constructed by this *method approximate the desired solution better than hypergeometric functions. The proposed mathematical model has the following advantages:*

- the given solution has the form of a polynomial of low degree, convenient for analysis, is obtained by simple means and does not require the involvement of a rather complex theory of hypergeometric functions;

- such a solution does not require summing the series and evaluating its remainder, which when using hypergeometric functions leads to additional difficulties. The solution in hypergeometric functions is a series that quickly converges only in the vicinity of a particular point of the interval. Therefore, solutions of different forms are constructed for different points. The proposed method approximates the desired function uniformly over the entire interval of change of the independent variable;

- the Lanczos method used allows one to construct polynomials that approximate the desired solution of the differential equation approximately times better than the partial sums of the 2^n*-th order of the Taylor series, which are used in the problem under consideration to represent hypergeometric functions.*

***Keywords**: τ-method, canonical Lanczos polynomials, stress state, creep, rheological behavior.*

Числові експериментальні дослідження, проведені у світі, показали, що в бетонних та залізобетонних конструкціях, що знаходяться під тривалою дією навантажень, виникають непружні деформації, які в кілька разів можуть перевищувати початкові пружні деформації. Тому питання проектування тривалого деформування бетону в часі є актуальним та має важливе значення.

У статті наводиться математична модель опису повзучості старіючого бетону за допомогою канонічних поліномів і методу Ланцоша. Наведений розв'язок задачі реологічної поведінки старіючого бетону враховує як ефекти спадкового характеру і зміну в часі модуля пружності, так і вплив віку матеріалу до моменту навантаження на граничні деформації повзучості.

Запропонована математична модель має наступні переваги:

1) поданий розв'язок має форму полінома невисокого степеня, зручний для аналізу і досягається простими засобами та не вимагає залучення досить складної теорії гіпергеометричних функцій;

2) такий розв'язок не пов'язаний із завданням підсумовування ряду та оцінкою його залишку, що при використанні гіпергеометричних функцій, не табульованих для ряду значень параметрів, викликає додаткові труднощі. Розв'язок в гіпергеометричних функціях являє собою ряд, що швидко збігається лише в околі тієї чи іншої точки інтервалу. Тому для різних точок будуються різні за формою розв'язки. Метод Ланцоша рівносильний пошуку розв'язку у формі розвинення за поліномами Чебишова. При цьому досягається рівномірна збіжність на інтервалі в порівнянні з використанням звичайних степеневих рядів.

3) застосований метод Ланцоша дозволяє побудувати поліноми, які наближають шуканий розв'язок диференціального рівняння до точного приблизно в 2^n *разів краще, ніж часткові суми n-го порядку ряду Тейлора, якими в розглянутій задачі і є гіпергеометричні функції.*

Практичне значення роботи полягає в тому, що запропонований підхід можна використовувати до розв'язування подібних задач.

***Ключові слова**: τ-метод, канонічні поліноми Ланцоша, напружений стан, повзучість, реологічна поведінка.*

Problem's Formulation

No structure can do without the use of the concrete today. The strength properties of the concrete and the ease of manufacture make it a unique material. It is primarily an artificial stone, similar in characteristics to granite and marble. The concrete and reinforced concrete are a complex composite of solid particles of silica, alumina, hematite, magnetite and limonite. These particles are interconnected by a layer of calcium carbonate molecules. Such a composite material has a fairly high strength.

Reliability and durability of the concrete and reinforced concrete structures and buildings cannot be ensured without full consideration of the features of concrete deformation. Numerous experimental studies conducted in the world have shown that in concrete and reinforced concrete structures under prolonged loads, inelastic deformations occur that can exceed the initial, instantaneous (elastic) deformations by several times. Therefore, the problem of prosthetics during prolonged deformation of concrete over time is relevant and important. The ability of concrete to deform over time under prolonged exposure to constant load is called creep. There are hypotheses that consider the mechanism of creep deformations under the influence of external load. [1]. The analysis of various types of deformations under load in concrete can be performed using rheological models. Such models that include ideal springs, dampers, valves, and other elements without revealing the physical mechanism of creep, offer its phenomenological description. [2].

One of the main factors affecting the creep of concrete is the relative humidity of the environment. Drying of specimens leads to an increase in the creep of concrete at an early age. When establishing a hygroscopic equilibrium between the environment and concrete before loading the specimens, the influence of relative humidity of the air is significantly less pronounced. Alternate wetting and drying of concrete increases the amount of creep deformation.

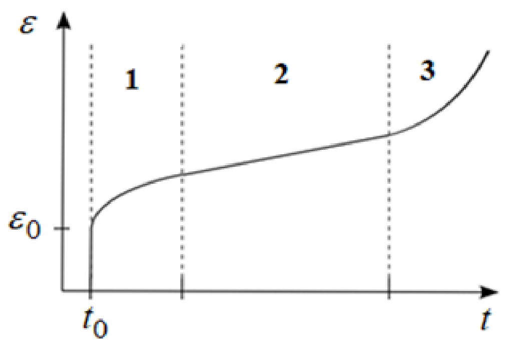

Fig. 1. A typical creep curve

A typical creep curve is presented in the figure (Fig. 1). Along the abscissa axis is the elapsed time from the start of the experiment, along the ordinate axis is the deformation at constant stress. The vertical segment with height ε_0 determines the instantaneous deformation at the moment t_0 of load application. Fragment **1** of the curve shows the increase in strain over time under constant load. It is clear that the strain rate that is equal to the derivative $d\varepsilon/dt$, decreases. This is a fragment of unsteady creep, after which steady creep (fragment **2**) occurs with a constant velocity $d\varepsilon/dt = const$, that significantly depends on the applied stress. Fragment **3** is characterized by an increase in the deformation rate and ends with the destruction of the sample [3].

The aging process of structural materials leads to changes in the physical and mechanical properties of both individual parts and the entire object under difficult operating conditions. Failure of any of the parts of a complex structure can lead to serious emergencies. The aging processes have been studied in detail before; studies have established that the aging of structural materials in engineering leads to an increase in strength, coercive force, electrical resistance, heat resistance, etc. However, sometimes the positive effect of aging causes undesirable phenomena: fragility increases, deform of parts appears, therefore, noticeable internal stresses arise due to «big aging» [2].

There are a large number of methods for calculating the behavior of elastoplastic materials that allow modeling the operation of structures and taking into account nonlinear and rheological properties of concrete today. Such calculations are usually associated with serious and complex calculations.

This paper presents a new solution to the problem of rheological behavior of an aging concrete body, that takes into account both the effects of hereditary nature and the change in the elastic modulus over time, as well as the influence of the age of the material before loading on the ultimate creep deformations using canonical polynomials and the Lanczos method. The polynomials constructed by this method approximate the desired solution better than hypergeometric functions.

Analysis of recent research and publications

The reliability and durability of concrete and reinforced concrete structures cannot be ensured without considering the important deformation properties of concrete. The rheological properties of concrete, primarily due to its creep, have a significant impact on the stress-strain state of the structure over time, even under constant external load. Over time, forces are redistributed between heavily and lightly loaded elements, and between the reinforcement and concrete in the cross-sections of the elements.

Many scientists have attempted to derive a formula for calculating creep strains and reflect the relationship between stresses and strains in an elastic-creeping medium based on the phenomenological equations of the mechanical state of the material. Among them are G.N. Maslov, N.Kh. Arutyunyan, I.E. Prokofievich, A.Ya. Barashikov, A.M. Bambura, A.B. Golyshev, O.I. Golodnov, B.G. Demchina, A.R. Rzhanitsyn, O.Ya. Berg, and many others. Significant contributions to the development of nonlinear creep theory were also made by A.G. Tamrozyan, A.M. Neville, V.M. Bondarenko, P.I. Vasiliev, and others.

Numerous foreign scientists have devoted their work to the issue of thermal creep: S. Bazant, R. Wendner, M. Hubler, and many others. Vast studies have also been published on the effect of concrete moisture content on the rate of plastic deformation of structures. Large studies and publications by the American Concrete Institute are devoted to methods for calculating concrete structures.

Formulation of the study purpose

The standard apparatus of continuum mechanics is used to study the motion of a medium under creep, which involves considering the stress, strain, and strain rate tensor. Generally, the displacement of a body's points can be significant compared to its initial dimensions. In this case, its deformations must be described using the finite strain tensor, and the corresponding boundary conditions must also be specified on the deformed surface of the body. However, the formulas obtained in this way for determining the crack formation moment are labor-intensive and inconvenient for practical use. V.V. Mikhailov suggests using tables and nomograms to facilitate calculations. A.A. Tamarin simplified the solution to this problem somewhat, but he, too, could not avoid tables and graphs of the states of bodies. A.A. Gvozdev and S.A. Dmitriev first proposed a method for determining the moment of crack formation using core moments to address these shortcomings. This approach is based on the principles of the resistance of elastic materials. It was the first and one of the simplest attempts to use core moments to calculate crack formation.

Presenting main material

The authors of the article propose a model of the rheological behavior of concrete based on the theory of hereditary aging that takes into account both the effects of hereditary nature and the change in the elastic modulus over time, as well as the influence of the material age before loading on the ultimate creep deformations. The relationship between stress $\sigma(t)$ and strain $\varepsilon(t)$ in the case of uniaxial compression is taken as

$$\sigma(t) = E_0^{-1}\left[\xi(t)\sigma(t) + \int_{t_0}^{1} \eta(s)H(t-s)\sigma(s)ds\right], \quad (1)$$

where $E(t) = E_0\xi^{-1}(t)$ is an instantaneous time-dependent elastic modulus; $\eta(t)$ and $\xi(t)$ is the monotonically decreasing functions characterizing the aging of the instantaneous and hereditary response of the material, respectively; $H(t-s)$ is a function with weak singularity of Abelian type.

Let $H(t-s)$ be the Rzhanitsyn function [5]

$$H(t-s) = \frac{(t-s)^{\alpha}}{\Gamma(r)} e^{\beta(t-s)}, \quad (2)$$

where $\Gamma(r)$ is a gamma function; $-1 < \alpha < 0$, $r = 1+\alpha$, $\beta < 0$ is the heredity parameters.

When chosen $\eta(s)$ in the form

$$\eta(s) = \chi(t_0)(1+\lambda e^{-\gamma s}) \quad \text{at} \quad \alpha > 0, \quad \gamma > 0, \quad (3)$$

then the kernel of equation (1) represents a positive, monotonically decreasing function, when $t-s$ increases, that asymptotically approaches zero. The influence function $\eta(s)H(t-s)$, when s increases, approaches a time-invariant kernel, i.e. characterizes a material in which the aging effect disappears over time. The multiplier $\chi(t_0)$ that determines the ultimate creep strain depends on the loading moment. and, as experiments show [3], this multiplier decreases, when t_0 increases. The change in material properties over time is also taken into account using the increasing modulus of elasticity $E(t)$. The rheological parameters included in the kernel of equation (1) can be determined by processing the three creep rate curves

$$\chi(T)=\sigma^{-1}\varepsilon(T+t_0)-E^{-1}(T+t_0),\quad T=t-t_0, \tag{4}$$

that are built on experimental dependencies $E(t)$ and $\varepsilon(t)$ at $\sigma=\sigma_0=const$, that corresponding to loads at three different ages of the material t_{oi}, $i=1,2,3$. The following describes the methodology for determining the rheological parameters that included in equation (1).

When $\sigma=\sigma_0=const$ in the equation (1), considering the expressions (2) and (3), for creep $\chi(T)$ we have

$$\chi(T)=T_0{}^{T}\left(1+le^{-g(t_0+s)}\right)\frac{(T-s)^{\alpha}}{\Gamma(r)}e^{b(T-a)}. \tag{5}$$

Applying the Laplace transform in a variable T to equation (5), we get an algebraic equation that contains the parameter p and requested rheological parameters

$$\chi^{L}(p)=\frac{\chi(t_0)}{(p-b)^{\Gamma}}\left[\frac{1}{p}+\frac{l}{p+g}e^{-gt_0}\right]. \tag{6}$$

The Laplace transform $\chi^{L}(p)$ for the function $c(T)$ for a fixed parameter value is determined by numerical integration using the quadratic interpolation formula [4]

$$p\chi^{L}(p)=T_0^{\Gamma}\chi\left[\frac{q}{p}\right]e^{-q}dq\sum_{k=1}^{n}A_k\chi\left[\frac{q_k}{p}\right]. \tag{7}$$

We obtain from equation (6) according to the limit theorem of operational calculus

$$\chi(\infty)=\lim_{p\to 0}\chi^{L}(p)\cdot p=\frac{\chi(t_0)}{(-\beta)^{\Gamma}}. \tag{8}$$

On the other hand, the limiting value $\chi(\infty)$ can be obtained from the dependence graph $c(t)$. Using three curves $c_i(t)$ corresponding to the loads at age t_{oi}, ($i=1,2,3$), we form three relations of the form (8). Let us add to them three more equations of the form (6), in which the value $p=p_1>0$ is chosen arbitrarily for reasons of convenience of processing the curves. From the obtained relations we find the equation for determining the parameter g

$$\Delta g_{12}\exp(-\gamma\Delta t_{32})+\Delta g_{23}\exp(-\gamma\Delta t_{12})=\Delta g_{13}, \tag{9}$$

where $g_k=\dfrac{c\,k^{L}(p_1)}{c\,k(\Gamma\ )}$, $\Delta g_{ik}=g_i-g_k$, $\Delta t_{ik}=t_{oi}-g_{ok}$, $i,k=1,2,3$. The solution to equation (9) can be obtained numerically.

We obtain the formula for the parameter l

$$\lambda=\frac{(1+\gamma)\Delta g_{12}}{g_2\exp(-\gamma t_{01})-g_1\exp(-\gamma t_{02})}. \tag{10}$$

From equations of the form (6) formed for two different values of the parameter $p=p_1$, $p=p_2$ $T_{n+1}*(q^{r})$ that correspond to the same function, for example $c_i(t)$, we find

$$b=\frac{p_1}{1-\left[1-p_2/b\right]^{w}},\qquad r=\frac{\ln h_1(p_1)}{\ln\left[1-p_1/b\right]^{w}}, \tag{11}$$

where $w=\dfrac{\ln h_1(p_1)}{\ln h_1(p_2)}$, $h_1(p_k)=g_1^{-1}(p_k)\left[\dfrac{1}{p_k}+\dfrac{l}{p_k+g}e^{-gt_{01}}\right]$, $k=1,2.$

The parameter $c(t_0)$ that characterizes the function of the ultimate deformation on the age of the material at the moment of loading is determined by the relation (8). Processing a series of simple concrete creep curves ($\sigma = \sigma_0 = const$) [5] with load moments $t_{01} = 5$, $t_{02} = 7$, $t_{03} = 28$ day gave the following parameter values: $g = 0{,}233$ day^{-1}, $l = 5{,}72$, $r = 0{,}614$, $b = -0{,}00838$ day^{-1}, $c(t_{01}) = 0{,}0185$, $c(t_{02}) = 0{,}017$, $c(t_{03}) = 0{,}00874$ day^{-2}.

Let's consider an approximate way to solve the equation (1). Taking into account expressions (2) and (3), this equation can be written in the form

$$\varepsilon(t)E_0 = \xi(t)\sigma(t) + e^{\beta t} I_\alpha * \eta(t) e^{-\beta t}\sigma(t). \qquad (12)$$

Here $I_a *$ is an integral operator acting on time functions

$$I_\alpha * f(t) = \int_{t_0}^{t} \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} f(s)ds. \qquad (13)$$

Functions $\varepsilon(t)$ and $\eta(t)$ decay over time, starting from a certain time t_1, become practically constant values. They can be approximated with sufficient accuracy by polynomials of low degree in interval $[t_0, t_1]$

$$\xi(t) = \sum_{m=0}^{1} \xi_m \theta^{rm}, \quad \eta(t) = \sum_{m=0}^{1} \eta_m \theta^{rm}, \qquad (14)$$

where $q = (t - t_0)/(t_1 - t_0)$.

Assuming that these functions have bounded derivatives, the approximations are also valid

$$E_0\varepsilon(t) = E_0 \sum_{m=0}^{n} \varepsilon_m \theta^{rm}, \qquad \sigma(t) = E_0 \sum_{m=0}^{n} \sigma_m \theta^{rm}. \qquad (15)$$

Let us substitute expressions (14) and (15) into equation (12) and introduce into the right-hand side magnitude of error (discrepancy) in the form, using Lanczos method

$$T_{n+1} * (q^r)(t_0 + t_1 q^r + ... + t_i q^{ri}), \qquad (16)$$

where $T_{n+1} * (\theta^r)$ is a shifted Chebyshev polynomial; t_k is the still unknown parameters.

We form a system of equations by equating the coefficients at the same powers of the variable q. Then we find all the unknowns $\sigma_m (m = 0,1, \ldots, n)$ and $\tau_k (k = 0,1, ..., n)$ from it. In this case, the requested function $\sigma(t)$ in an expression (15) is defined as an expansion in powers q.

The Lanczos method is equivalent to finding a solution in the form of an expansion by Chebyshev polynomials. In this case, faster uniform convergence on the interval is achieved than when using ordinary power series. The error of the solution is characterized by the magnitude of the discrepancy, which, due to the property of Chebyshev polynomials, does not exceed the sum $|t_0| + |t_1| + ... + |t_n|$.

Conclusions

The mathematical model of creep of aging concrete using canonical polynomials is considered and Lanczos τ-method in the article. The resulting solution has the form of a polynomial of low degree (most often the 2nd degree is sufficient); it is convenient for analysis and does not require the involvement of a rather complex theory of hypergeometric functions. In addition, the presented method allows you to approximate the desired solution of the differential equation approximately 2^n times better than other methods. The proposed solution approximates the requested function uniformly over the entire interval of change of the independent variable. Canonical polynomials (in general form or numerically), defined by the relation derived in this article, can be constructed for other rheology problems. The method takes into account both the effects of hereditary nature and the change in the elastic modulus over time, as well as the influence of the age of the material before the moment of loading on the creep limit deformations.

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